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Cubic curves from matrix models and generalized Konishi anomalies

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Abstract: We study the matrix model/gauge theory connection for three different $\mathcal{N} = 1$ models: $\mathrm{U}(N)\times\mathrm{U}(N)$ with matter in bifundamental representations, $\mathrm{U}(N)$ with matter in the symmetric representation, and $\mathrm{U}(N)$ with matter in the antisymmetric representation. Using Ward identities, we explicitly show that the loop equations of the matrix models lead to cubic algebraic curves. We then establish the equivalence of the matrix model and gauge theory descriptions in two ways. First, we derive generalized Konishi anomaly equations in the gauge theories, showing that they are identical to the matrix-model equations. Second, we use a perturbative superspace analysis to establish the relation between the gauge theories and the matrix models. We find that the gauge coupling matrix for $\mathrm{U}(N)$ with matter in the symmetric or antisymmetric representations is *not* given by the second derivative of the matrix-model free energy. However, the matrix-model prescription can be modified to give the gauge coupling matrix.

Keywords: Nonperturbative Effects, 1/N Expansion, Extended Supersymmetry, Matrix Models.

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1. Introduction

The matrix model approach [1] has provided a new way of studying (the holomorphic sector of) supersymmetric gauge theories. That the matrix model leads to results identical to those of the gauge theory has been shown for the simplest model (U(N) with adjoint matter) using two methods. First, a remarkably succinct perturbative superspace argument was used to show [2] that the effective superpotential is equal to the corresponding matrix-model quantity order-by-order in a perturbative expansion in powers of the glueball field. Second, it was shown [3] that the (quadratic) loop equation of the matrix model is realized in the chiral ring of the gauge theory as a generalization of the Konishi anomaly equation [4], thus establishing the (non-perturbative) correctness of the matrix-model description. The latter method was extended to include fundamental matter in ref. [5]. The perturbative method can also be used to treat this case, although it was treated in less detail in ref. [2]. Some related earlier work and more recent developments can be found in refs. [6, 7].

In this work, we extend the matrix model/gauge theory equivalence to three $\mathcal{N} = 1$ theories:[1] U(N)× U(N) gauge theory with matter in adjoint and bifundamental representations, U(N) gauge theory with matter in the adjoint and symmetric representations, and

[1] These theories have in common that, in the $\mathcal{N} = 2$ limit, they all possess non-hyperelliptic (cubic) Seiberg-Witten curves.

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U(N) gauge theory with matter in the adjoint and antisymmetric representations. We derive the cubic relations

$$u^3 - r(z)\, u - s(z) = 0\,, \tag{1.1}$$

satisfied by the resolvents of the associated matrix models, and give explicit expressions for the coefficients of the polynomials $r(z)$ and $s(z)$ in terms of the adjoint-field eigenvalues, using a Ward-identity approach. These loop equations encode the geometry of cubic algebraic curves underlying these models. On the gauge theory side we consider generalized Konishi anomaly equations and show that they lead to equations identical to the matrix-model loop equations, thus establishing the equivalence. We also use a perturbative superspace analysis to analyze the relation between the gauge theories and the matrix models. We find that for the U(N) models with matter in the symmetric/antisymmetric representations, the gauge coupling matrix is *not* given by the second derivative of the matrix-model free energy. Nevertheless, the matrix-model prescription can be modified to give the gauge theory coupling matrix.

Various aspects of the U(N)$\times$ U(N) model were discussed in refs. [8]–[10] and also recently in ref. [11]. The U(N) models with symmetric or antisymmetric matter were also studied recently in ref. [12]. There is some overlap between the present work and the recent papers [11, 12], but for the most part our work is complementary to their analysis. The explicit expressions for $s(z)$ that we derive in this paper were also obtained in [11, 12] (using a different method); however, the gauge theory analogs of the loop equations were not discussed and the equivalence was not established.

In section 2, we discuss the supersymmetric U(N)$\times$ U(N) theory with bifundamental matter. In section 3, we perform a similar analysis for the supersymmetric U(N) gauge theory with matter in symmetric or in antisymmetric representations. In section 4, we use superspace perturbation theory to analyze the U(N) models. A summary of the main results of the paper can be found in section 5. In the appendices, we briefly discuss the saddle-point approach as an alternative to the method used in the main text, and collect some background material on the relevant representations.

2. U(N)$\times$ U(N) with bifundamental matter

In this section we study the $\mathcal{N} = 1$ U(N)$\times$ U(N) supersymmetric gauge theory with the following matter content: two chiral superfields ${\phi_i}^j$, ${\tilde\phi_{\tilde\imath}}{}^{\tilde\jmath}$ transforming in the adjoint representation of each of the two factors of the gauge group, one chiral superfield ${b_i}^{\tilde\jmath}$ transforming in the bifundamental representation $(\square, \bar\square)$, and one chiral superfield ${\tilde b_{\tilde\imath}}{}^j$ transforming in the bifundamental representation $(\bar\square, \square)$. The superpotential of the gauge theory is taken to be of the form[2]

$$\mathcal{W}(\phi, \tilde\phi, b, \tilde b) = \mathrm{tr}\,[W(\phi) - \tilde W(\tilde\phi) - \tilde b\, \phi\, b + b\, \tilde\phi\, \tilde b]\,, \tag{2.1}$$

[2] An explicit mass term for the bifundamental field, $m\,\mathrm{tr}(b\tilde b)$, can be introduced by shifting ϕ and $\tilde\phi$ and redefining the coefficients in $W(\phi)$ and $\tilde W(\tilde\phi)$; to simplify the presentation we will therefore not explicitly include such a term, although we think of the bifundamental fields as being massive.

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where $W(\phi) = \sum_{m=1}^{N+1}(g_m/m)\,\phi^m$ and similarly for $\tilde{W}(\tilde{\phi})$. This superpotential can be viewed as a deformation of an $\mathcal{N}=2$ theory.

Below, after a detailed derivation of the loop equations of the matrix model, we establish the non-perturbative equivalence between the holomorphic sector of the above gauge theory and the associated matrix model, following the ideas developed in ref. [3]. (The argument for the perturbative equivalence of the matrix model and gauge theory given in [2] goes through essentially unchanged for this case.) More precisely, we show that the matrix-model loop equations are encoded in the gauge theory as vacuum expectation values of divergences of certain anomalous currents. The anomalies associated with these currents are generalizations of the Konishi anomaly [4].

2.1 Matrix model analysis

Following the ideas of Dijkgraaf and Vafa, we take the partition function for the matrix model associated with the above gauge theory to be[3]

$$Z = \int \mathrm{d}\Phi\,\mathrm{d}\tilde{\Phi}\,\mathrm{d}B\,\mathrm{d}\tilde{B}\,\exp\left(-\frac{1}{g_{\mathrm{s}}}\mathrm{tr}\Big[W(\Phi)-\tilde{W}(\tilde{\Phi})-\tilde{B}\Phi B+B\tilde{\Phi}\tilde{B}\Big]\right), \tag{2.2}$$

where Φ is an $M\times M$ matrix, $\tilde{\Phi}$ is an $\tilde{M}\times\tilde{M}$ matrix, B is an $M\times\tilde{M}$ matrix, and $\tilde{B}$ is an $\tilde{M}\times M$ matrix. These matrices should be viewed as holomorphic quantities [13, 3, 11] and the integrals in (2.2) are along some curve. This point was emphasized in the recent paper [11], where the above model was also studied. We are interested in the planar limit of the matrix model, i.e. the limit in which $g_{\mathrm{s}}\to 0$ and M, $\tilde{M}\to\infty$, keeping $S=g_{\mathrm{s}}M$ and $\tilde{S}=g_{\mathrm{s}}\tilde{M}$ fixed.

In the saddle-point approach to this model [14, 15, 8, 10, 11], one diagonalizes the matrices Φ and $\tilde{\Phi}$, and derives equations satisfied by the resolvents[4]

$$\omega(z) = g_{\mathrm{s}}\left\langle \mathrm{tr}\left(\frac{1}{z-\Phi}\right)\right\rangle = g_{\mathrm{s}}\sum_i \frac{1}{z-\lambda_i}\,; \quad \tilde{\omega}(z) = g_{\mathrm{s}}\left\langle \mathrm{tr}\left(\frac{1}{z-\tilde{\Phi}}\right)\right\rangle = g_{\mathrm{s}}\sum_i \frac{1}{z-\tilde{\lambda}_i}\,, \tag{2.3}$$

where matrix-model expectation values are defined via

$$\left\langle \mathcal{O}(\Phi,\tilde{\Phi},B,\tilde{B})\right\rangle \equiv \frac{1}{Z}\int \mathrm{d}\Phi\,\mathrm{d}\tilde{\Phi}\,\mathrm{d}B\,\mathrm{d}\tilde{B}\,\mathcal{O}(\Phi,\tilde{\Phi},B,\tilde{B})\,\mathrm{e}^{-\frac{1}{g_{\mathrm{s}}}\mathrm{tr}[W(\Phi)-\tilde{W}(\tilde{\Phi})-\tilde{B}\Phi B+B\tilde{\Phi}\tilde{B}]}\,. \tag{2.4}$$

For completeness we give some details of the saddle-point approach in appendix A.

Below we derive the equations satisfied by the resolvents using an approach [3] that is close in spirit to the gauge theory analysis given in section 2.2. (We stress that this method does not assume that the matrices are hermitean.)

Throughout the paper we often suppress matrix indices, assuming that multiplications are done using the natural contractions.

[3]We use capital letters to denote matrix model quantities.

[4]We use an unconventional normalization of the resolvents in order to make the comparison with gauge theory more transparent. Also, in order not to clutter the formulæ we drop the $\langle\cdots\rangle$ when writing expressions in terms of eigenvalues.

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Quadratic relations. We start by considering the Ward identity

$$\begin{aligned} 0 &= \frac{g_{\rm s}^2}{Z}\int {\rm d}\Phi\,{\rm d}\tilde\Phi\,{\rm d}B\,{\rm d}\tilde B\,\frac{\rm d}{{\rm d}B_i{}^{\tilde j}}\left\{\left(\frac{1}{z-\Phi}B\frac{1}{z-\tilde\Phi}\right)_i{}^{\tilde j}\,{\rm e}^{-\frac{1}{g_{\rm s}}{\rm tr}[W(\Phi)-\tilde W(\tilde\Phi)-\tilde B\Phi B+B\tilde\Phi\tilde B]}\right\} \\ &= g_{\rm s}^2\left\langle {\rm tr}\left(\frac{1}{z-\Phi}\right){\rm tr}\left(\frac{1}{z-\tilde\Phi}\right)\right\rangle + g_{\rm s}\left\langle {\rm tr}\left(\frac{\Phi}{z-\Phi}B\frac{1}{z-\tilde\Phi}\tilde B\right)\right\rangle - g_{\rm s}\left\langle {\rm tr}\left(\frac{1}{z-\Phi}B\frac{\tilde\Phi}{z-\tilde\Phi}\tilde B\right)\right\rangle \\ &= \omega(z)\,\tilde\omega(z) - g_{\rm s}\left\langle {\rm tr}\left(B\frac{1}{z-\tilde\Phi}\tilde B\right)\right\rangle + g_{\rm s}\left\langle {\rm tr}\left(\tilde B\frac{1}{z-\Phi}B\right)\right\rangle, \end{aligned} \tag{2.5}$$

where the resolvents were defined in (2.3), and we have used (here and throughout) the factorization of expectation values in the planar limit. Thus, the expectation values and resolvents appearing in (2.5) (and in all remaining equations in this section) refer only to the planar (sphere) parts in the genus expansion; we will not indicate this explicitly as confusion is unlikely to arise.

Next, we note that for any polynomial $f(z)$, we have the Ward identity

$$\begin{aligned} 0 &= \frac{g_{\rm s}^2}{Z}\int {\rm d}\Phi\,{\rm d}\tilde\Phi\,{\rm d}B\,{\rm d}\tilde B\,\frac{\rm d}{{\rm d}\Phi_i{}^{j}}\left\{\left(\frac{f(\Phi)}{z-\Phi}\right)_i{}^{j}\,{\rm e}^{-\frac{1}{g_{\rm s}}{\rm tr}[W(\Phi)-\tilde W(\tilde\Phi)-\tilde B\Phi B+B\tilde\Phi\tilde B]}\right\} \\ &= \omega(z)^2 f(z) - g_{\rm s}^2\left\langle {\rm tr}\left[\frac{\rm d}{{\rm d}\Phi}\left(\frac{f(z)-f(\Phi)}{z-\Phi}\right)\right]\right\rangle - g_{\rm s}\left\langle {\rm tr}\left(\frac{f(\Phi)W'(\Phi)}{z-\Phi}\right)\right\rangle + \\ &\quad + g_{\rm s}\left\langle {\rm tr}\left(\tilde B\frac{f(\Phi)}{z-\Phi}B\right)\right\rangle. \end{aligned} \tag{2.6}$$

In particular, setting $f(\Phi)=1$, eq. (2.6) simplifies to

$$\omega(z)^2 - W'(z)\,\omega(z) = -g_{\rm s}\left\langle {\rm tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\right)\right\rangle - g_{\rm s}\left\langle {\rm tr}\left(\tilde B\frac{1}{z-\Phi}B\right)\right\rangle. \tag{2.7}$$

For future reference we also note that by multiplying eq. (2.7) by $f(z)$ and combining the resulting expression with eq. (2.6) one finds

$$g_{\rm s}\left\langle {\rm tr}\left(\tilde B\frac{f(z)-f(\Phi)}{z-\Phi}B\right)\right\rangle = -g_{\rm s}^2\left\langle {\rm tr}\left[\frac{\rm d}{{\rm d}\Phi}\left(\frac{f(z)-f(\Phi)}{z-\Phi}\right)\right]\right\rangle + g_{\rm s}\left\langle {\rm tr}\left(\frac{f(z)-f(\Phi)}{z-\Phi}W'(\Phi)\right)\right\rangle. \tag{2.8}$$

Analogously, one can show

$$0=\tilde\omega(z)^2 f(z) - g_{\rm s}^2\left\langle {\rm tr}\left[\frac{\rm d}{{\rm d}\tilde\Phi}\left(\frac{f(z)-f(\tilde\Phi)}{z-\tilde\Phi}\right)\right]\right\rangle + g_{\rm s}\left\langle {\rm tr}\left(\frac{f(\tilde\Phi)\tilde W'(\tilde\Phi)}{z-\tilde\Phi}\right)\right\rangle - g_{\rm s}\left\langle {\rm tr}\left(B\frac{f(\tilde\Phi)}{z-\tilde\Phi}\tilde B\right)\right\rangle, \tag{2.9}$$

from which it follows by setting $f(\tilde\Phi)=1$ that

$$\tilde\omega(z)^2 + \tilde W'(z)\,\tilde\omega(z) = g_{\rm s}\left\langle {\rm tr}\left(\frac{\tilde W'(z)-\tilde W'(\tilde\Phi)}{z-\tilde\Phi}\right)\right\rangle + g_{\rm s}\left\langle {\rm tr}\left(B\frac{1}{z-\tilde\Phi}\tilde B\right)\right\rangle. \tag{2.10}$$

By combining the previous two equations, we obtain

$$g_{\rm s}\left\langle {\rm tr}\left(B\frac{f(z)-f(\tilde\Phi)}{z-\tilde\Phi}\tilde B\right)\right\rangle = g_{\rm s}^2\left\langle {\rm tr}\left[\frac{\rm d}{{\rm d}\tilde\Phi}\left(\frac{f(z)-f(\tilde\Phi)}{z-\tilde\Phi}\right)\right]\right\rangle + g_{\rm s}\left\langle {\rm tr}\left(\frac{f(z)-f(\tilde\Phi)}{z-\tilde\Phi}\tilde W'(\tilde\Phi)\right)\right\rangle. \tag{2.11}$$

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From the above equations it is possible to derive a quadratic relation among the resolvents that does not involve expectation values with B, $\tilde{B}$'s. Combining eqs. (2.5), (2.7), and (2.10) to eliminate the B-dependent terms, one obtains the following quadratic relation involving the two resolvents

$$\omega(z)^2 + \tilde{\omega}(z)^2 - \omega(z)\,\tilde{\omega}(z) - W'(z)\,\omega(z) + \tilde{W}'(z)\,\tilde{\omega}(z) = r_1(z)\,, \tag{2.12}$$

where

$$\begin{aligned} r_1(z) &= -g_{\rm s}\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\right)\right\rangle + g_{\rm s}\left\langle \mathrm{tr}\left(\frac{\tilde{W}'(z)-\tilde{W}'(\tilde{\Phi})}{z-\tilde{\Phi}}\right)\right\rangle \\ &= -g_{\rm s}\sum_i \frac{W'(z)-W'(\lambda_i)}{z-\lambda_i} + g_{\rm s}\sum_i \frac{\tilde{W}'(z)-\tilde{W}'(\tilde{\lambda}_i)}{z-\tilde{\lambda}_i}\,, \end{aligned} \tag{2.13}$$

is a polynomial of degree at most $N-1$.

The cubic algebraic curve. We now discuss how the cubic algebraic curve that underlies the model [8, 10] emerges. One can eliminate the terms linear in the resolvents in eq. (2.12) by defining

$$\omega(z) = u_1(z) + \omega_r(z)\,, \qquad \tilde{\omega}(z) = -u_3(z) + \tilde{\omega}_r(z)\,, \tag{2.14}$$

where

$$\omega_r(z) = \frac{2}{3}W'(z) - \frac{1}{3}\tilde{W}'(z)\,, \qquad \tilde{\omega}_r(z) = \frac{1}{3}W'(z) - \frac{2}{3}\tilde{W}'(z)\,, \tag{2.15}$$

giving

$$u_1(z)^2 + u_3(z)^2 + u_1(z)u_3(z) = r_0(z) + r_1(z) = r(z)\,, \tag{2.16}$$

with

$$\begin{aligned} r_0(z) &= \omega_r^2(z) + \tilde{\omega}_r^2(z) - \omega_r(z)\tilde{\omega}_r(z) \\ &= \frac{1}{3}\left[W'^2(z) + \tilde{W}'^2(z) - W'(z)\tilde{W}'(z)\right]\,, \end{aligned} \tag{2.17}$$

a polynomial of degree $2N$.

Multiplying eq. (2.16) by $u_1(z) - u_3(z)$, one finds

$$u_1(z)^3 - r(z)\,u_1(z) = u_3(z)^3 - r(z)\,u_3(z) \equiv s(z)\,, \tag{2.18}$$

so that $u_1(z)$ and $u_3(z)$ are both roots of the cubic equation

$$0 = u^3 - r(z)\,u - s(z) = [u - u_1(z)][u - u_2(z)][u - u_3(z)]\,. \tag{2.19}$$

The absence of the quadratic term implies that the third root is $u_2(z) = -u_1(z) - u_3(z)$, so

$$s(z) = u_1(z)\,u_2(z)\,u_3(z) = [\omega(z) - \omega_r(z)][-\omega(z) + \tilde{\omega}(z) + \omega_r(z) - \tilde{\omega}_r(z)][-\tilde{\omega}(z) + \tilde{\omega}_r(z)], \tag{2.20}$$

which we will show to be a polynomial below.

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Defining $s(z) = s_0(z) + s_1(z)$ with

$$\begin{aligned} s_0(z) &= -\omega_r(z)\,\tilde{\omega}_r(z)\,[\omega_r(z) - \tilde{\omega}_r(z)] \\ &= \frac{1}{27}[-W'(z) + 2\tilde{W}'(z)][2W'(z) - \tilde{W}'(z)][W'(z) + \tilde{W}'(z)]\,, \end{aligned} \tag{2.21}$$

a polynomial of degree $3N$, we can rewrite the cubic equation (2.19) as

$$\begin{aligned} r_1(z)u + s_1(z) &= u^3 - r_0(z)u - s_0(z) \\ &= (u + \omega_r(z))(u - \omega_r(z) + \tilde{\omega}_r(z))(u - \tilde{\omega}_r(z))\,. \end{aligned} \tag{2.22}$$

From eqs. (2.20) and (2.21) it follows that

$$\begin{aligned} s_1(z) = {} & \omega(z)\,\tilde{\omega}(z)[\omega(z) - \tilde{\omega}(z)] - \frac{2}{3}[W'(z) + \tilde{W}'(z)]\omega(z)\,\tilde{\omega}(z) \\ & - \tilde{\omega}_r(z)\,[\omega(z)^2 - W'(z)\omega(z)] + \omega_r(z)\,[\tilde{\omega}(z)^2 + \tilde{W}'(z)\tilde{\omega}(z)]\,. \end{aligned} \tag{2.23}$$

We will show below that $s_1(z)$ is a polynomial of degree at most $2N-1$.

Cubic relations. Above we studied Ward identities leading to expressions with at most two bifundamental fields. We will now analyze expressions involving two additional bifundamental fields. The resulting equations can be used to derive a cubic relation among the resolvents of the form (2.23). The fact that one need not consider Ward identities with an even larger number of bifundamental fields can be traced to the form of the potential (2.1).

Our starting point is the Ward identity

$$\begin{aligned} 0 &= \frac{g_s^2}{Z}\int \mathrm{d}\Phi\,\mathrm{d}\tilde{\Phi}\,\mathrm{d}B\,\mathrm{d}\tilde{B}\,\frac{\mathrm{d}}{\mathrm{d}\tilde{\Phi}_{\tilde{\imath}}{}^{\tilde{\jmath}}}\left\{\left(\tilde{B}B\frac{1}{z-\tilde{\Phi}}\right)_{\tilde{\imath}}{}^{\tilde{\jmath}}\,\mathrm{e}^{-\frac{1}{g_s}\mathrm{tr}[W(\Phi)-\tilde{W}(\tilde{\Phi})-\tilde{B}\Phi B+B\tilde{\Phi}\tilde{B}]}\right\} \\ &= g_s\,\tilde{\omega}(z)\left\langle \mathrm{tr}\left(B\frac{1}{z-\tilde{\Phi}}\tilde{B}\right)\right\rangle + g_s\left\langle \mathrm{tr}\left(B\frac{\tilde{W}'(\tilde{\Phi})}{z-\tilde{\Phi}}\tilde{B}\right)\right\rangle - g_s\left\langle \mathrm{tr}\left(\tilde{B}B\frac{1}{z-\tilde{\Phi}}\tilde{B}B\right)\right\rangle. \end{aligned} \tag{2.24}$$

Similarly,

$$0 = g_s\,\omega(z)\left\langle \mathrm{tr}\left(\tilde{B}\frac{1}{z-\Phi}B\right)\right\rangle - g_s\left\langle \mathrm{tr}\left(\tilde{B}\frac{W'(\Phi)}{z-\Phi}B\right)\right\rangle + g_s\left\langle \mathrm{tr}\left(B\tilde{B}\frac{1}{z-\Phi}\tilde{B}B\right)\right\rangle. \tag{2.25}$$

We will also need

$$\begin{aligned} 0 = {} & \frac{g_s^2}{Z}\int \mathrm{d}\Phi\,\mathrm{d}\tilde{\Phi}\,\mathrm{d}B\,\mathrm{d}\tilde{B}\,\frac{\mathrm{d}}{\mathrm{d}\tilde{B}_{\tilde{\imath}}{}^{j}}\left\{\left(\frac{1}{z-\tilde{\Phi}}\tilde{B}B\tilde{B}\frac{1}{z-\Phi}\right)_{\tilde{\imath}}{}^{j}\,\mathrm{e}^{-\frac{1}{g_s}\mathrm{tr}[W(\Phi)-\tilde{W}(\tilde{\Phi})-\tilde{B}\Phi B+B\tilde{\Phi}\tilde{B}]}\right\} \\ = {} & g_s\,\tilde{\omega}(z)\left\langle \mathrm{tr}\left(\tilde{B}\frac{1}{z-\Phi}B\right)\right\rangle + g_s\,\omega(z)\left\langle \mathrm{tr}\left(B\frac{1}{z-\tilde{\Phi}}\tilde{B}\right)\right\rangle - g_s\left\langle \mathrm{tr}\left(\tilde{B}B\frac{1}{z-\tilde{\Phi}}\tilde{B}B\right)\right\rangle + \\ & + g_s\left\langle \mathrm{tr}\left(B\tilde{B}\frac{1}{z-\Phi}B\tilde{B}\right)\right\rangle. \end{aligned} \tag{2.26}$$

One can eliminate the terms quartic in the bifundamental fields from the above three equations; by combining (2.24), (2.25), and (2.26) and also using eq. (2.5), one finds

$$\omega(z)\,\tilde{\omega}(z)\,[\omega(z) - \tilde{\omega}(z)] = -g_s\left\langle \mathrm{tr}\left(\tilde{B}\frac{W'(\Phi)}{z-\Phi}B\right)\right\rangle + g_s\left\langle \mathrm{tr}\left(B\frac{\tilde{W}'(\tilde{\Phi})}{z-\tilde{\Phi}}\tilde{B}\right)\right\rangle. \tag{2.27}$$

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Finally, by using eqs. (2.5), (2.7), (2.8), (2.10), and (2.11) to eliminate the remaining dependence on the bifundamental fields, one obtains the cubic relation (2.23) with the following explicit expression for $s_1(z)$

$$
\begin{aligned}
s_1(z) \;=\;& g_{\rm s}\,\tilde\omega_r(z)\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\right)\right\rangle + g_{\rm s}\,\omega_r(z)\left\langle \mathrm{tr}\left(\frac{\tilde W'(z)-\tilde W'(\tilde\Phi)}{z-\tilde\Phi}\right)\right\rangle \\
& - g_{\rm s}^2\left\langle \mathrm{tr}\left[\frac{\mathrm{d}}{\mathrm{d}\Phi}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\right)\right]\right\rangle - g_{\rm s}^2\left\langle \mathrm{tr}\left[\frac{\mathrm{d}}{\mathrm{d}\tilde\Phi}\left(\frac{\tilde W'(z)-\tilde W'(\tilde\Phi)}{z-\tilde\Phi}\right)\right]\right\rangle \\
& + g_{\rm s}\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}W'(\Phi)\right)\right\rangle - g_{\rm s}\left\langle \mathrm{tr}\left(\frac{\tilde W'(z)-\tilde W'(\tilde\Phi)}{z-\tilde\Phi}\tilde W'(\tilde\Phi)\right)\right\rangle. \qquad (2.28)
\end{aligned}
$$

At this point, it is clear that $s_1(z)$ is a polynomial of degree at most $2N-1$, whose coefficients depend on the vevs $\langle \mathrm{tr}(\Phi^k)\rangle$ and $\langle \mathrm{tr}(\tilde\Phi^k)\rangle$ with $k \le 2N-1$.

We will now write $s_1(z)$ more explicitly, in terms of the eigenvalues λ_i and $\tilde\lambda_i$ of Φ and $\tilde\Phi$ respectively. First observe that, since $\frac{f(z)-f(\Phi)}{z-\Phi} \equiv \sum_m c_m \Phi^m$ is a polynomial, we have

$$
\begin{aligned}
\mathrm{tr}\left[\frac{\mathrm{d}}{\mathrm{d}\Phi}\left(\frac{f(z)-f(\Phi)}{z-\Phi}\right)\right] &= \sum_m c_m \sum_{k=0}^{m-1}\mathrm{tr}(\Phi^k)\mathrm{tr}(\Phi^{m-k-1}) = \sum_{i,j}\sum_m c_m \sum_{k=0}^{m-1}\lambda_i^k\lambda_j^{m-k-1} \\
&= \sum_{i,j}\sum_m c_m \frac{\lambda_i^m-\lambda_j^m}{\lambda_i-\lambda_j} \\
&= \sum_{i,j}\frac{1}{\lambda_i-\lambda_j}\left[\frac{f(z)-f(\lambda_i)}{z-\lambda_i}-\frac{f(z)-f(\lambda_j)}{z-\lambda_j}\right] \\
&= 2\sum_{i\neq j}\frac{1}{\lambda_i-\lambda_j}\left[\frac{f(z)-f(\lambda_i)}{z-\lambda_i}\right]. \qquad (2.29)
\end{aligned}
$$

Hence, we may write (suppressing $\langle\cdots\rangle$ in the eigenvalue basis)

$$
\begin{aligned}
s_1(z) \;=\;& g_{\rm s}\,\tilde\omega_r(z)\sum_i\left[\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i}\right] + g_{\rm s}\,\omega_r(z)\sum_i\left[\frac{\tilde W'(z)-\tilde W'(\tilde\lambda_i)}{z-\tilde\lambda_i}\right] - \\
& - 2g_{\rm s}^2\sum_{i\neq j}\frac{1}{\lambda_i-\lambda_j}\left[\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i}\right] - 2g_{\rm s}^2\sum_{i\neq j}\frac{1}{\tilde\lambda_i-\tilde\lambda_j}\left[\frac{\tilde W'(z)-\tilde W'(\tilde\lambda_i)}{z-\tilde\lambda_i}\right] + \\
& + g_{\rm s}\sum_i W'(\lambda_i)\left[\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i}\right] - g_{\rm s}\sum_i \tilde W'(\tilde\lambda_i)\left[\frac{\tilde W'(z)-\tilde W'(\tilde\lambda_i)}{z-\tilde\lambda_i}\right]. \qquad (2.30)
\end{aligned}
$$

Finally, using the saddle point equations (A.2), we may rewrite this as

$$
\begin{aligned}
s_1(z) \;=\;& g_{\rm s}\,\tilde\omega_r(z)\sum_i\left[\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i}\right] + g_{\rm s}\,\omega_r(z)\sum_i\left[\frac{\tilde W'(z)-\tilde W'(\tilde\lambda_i)}{z-\tilde\lambda_i}\right] - \\
& - g_{\rm s}^2\sum_{i,j}\frac{1}{\lambda_i-\tilde\lambda_j}\left[\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i}-\frac{\tilde W'(z)-\tilde W'(\tilde\lambda_j)}{z-\tilde\lambda_j}\right], \qquad (2.31)
\end{aligned}
$$

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which is a polynomial of degree no more that $2N-1$. This result also appeared recently in ref. [11], although using a different method of derivation.

This concludes our discussion of the $\mathrm{U}(N)\times \mathrm{U}(N)$ matrix model. We now turn to the gauge theory analysis.

2.2 Gauge theory analysis

As will now be shown, the matrix-model loop equations can be obtained from certain generalizations of the Konishi anomaly equations in the gauge theory. We find a one-to-one correspondence with the matrix-model formulæ derived above.

It is sufficient to study the chiral part of the anomaly equations [3], i.e. one may use identities that hold in the chiral ring. The chiral ring is defined as all chiral operators modulo terms of the form $\{\bar{Q}_{\dot\alpha},\cdot\}$. For a Grassmann even field F, one therefore has, in the chiral ring, $0=[\bar{Q}^{\dot\alpha},D_{\alpha\dot\alpha}F]=\mathsf{W}_\alpha F$, where W_α is the (spinor) gauge superfield. As discussed in appendix B, W_α can be viewed as a diagonal 2×2 matrix where the entries along the diagonal are the gauge superfields of the two $\mathrm{U}(N)$ factors, $\mathcal{W}_\alpha$ and $\tilde{\mathcal{W}}_\alpha$. More explicitly, $\mathsf{W}_\alpha=\mathsf{W}^A_\alpha T^A$, where T^A are the representation matrices appropriate for the action of the gauge field on F. Using the explicit expressions for T^A given in appendix B one obtains identities (in the chiral ring) among the adjoint and bifundamental fields of the form (as anticipated in [3])

$$[\mathcal{W}_\alpha,\phi]=0\,,\qquad [\tilde{\mathcal{W}}_\alpha,\tilde\phi]=0\,,\qquad \mathcal{W}_\alpha\, b=b\,\tilde{\mathcal{W}}_\alpha\,,\qquad \tilde{\mathcal{W}}_\alpha\,\tilde b=\tilde b\,\mathcal{W}_\alpha\,. \tag{2.32}$$

These identities will be freely used in what follows. The Grassmann oddness of $\mathcal{W}_\alpha$ together with the relations $\mathcal{W}_\alpha=\epsilon_{\alpha\beta}\mathcal{W}^\beta$ and $\epsilon^{\beta\alpha}\epsilon_{\alpha\gamma}=\delta^\beta_\gamma$ will also be used below.

The basic building blocks that we use are the anomalous currents $\bar{\tilde b}_{\tilde i}{}^j(e^{\mathsf{V}}b)_k{}^{\tilde l}$, $\bar\phi_i{}^j(e^{\mathsf{V}}\phi)_k{}^l$, and $\bar{\tilde\phi}_{\tilde i}{}^{\tilde j}(e^{\mathsf{V}}\tilde\phi)_{\tilde k}{}^{\tilde l}$, where V is the (vector) gauge superfield (see appendix B for more details about the notation).

We are interested in the action of $\bar D^2$ on the currents. Using the superpotential (2.1), one finds the classical piece of $\bar D^2\bar{\tilde b}_{\tilde i}{}^j(e^{\mathsf{V}}b)_k{}^{\tilde l}$ to be $[-(\tilde b\phi)_{\tilde i}{}^j+(\tilde\phi\tilde b)_{\tilde i}{}^j]b_k{}^{\tilde l}$. This current also has an anomaly (see appendix B for an explanation of the notation)

$$\begin{aligned}&\frac{1}{32\pi^2}(\mathsf{W}_\alpha)_N{}^M(\mathsf{W}^\alpha)_Q{}^P(T_M{}^N)_{\tilde i}{}^j{}_m{}^{\tilde n}(T_P{}^Q)_{\tilde n}{}^m{}_k{}^{\tilde l}= \\ &\quad=\frac{1}{32\pi^2}\left[(\tilde{\mathcal{W}}_\alpha\tilde{\mathcal{W}}^\alpha)_{\tilde i}{}^{\tilde l}\delta^j_k-(\mathcal{W}_\alpha)_k{}^j(\tilde{\mathcal{W}}^\alpha)_{\tilde i}{}^{\tilde l}-(\tilde{\mathcal{W}}_\alpha)_{\tilde i}{}^{\tilde l}(\mathcal{W}^\alpha)_k{}^j+(\mathcal{W}_\alpha\mathcal{W}^\alpha)_k{}^j\delta^{\tilde l}_{\tilde i}\right].\end{aligned} \tag{2.33}$$

There might be perturbative corrections to the anomaly but these will be non-chiral [3] and so are not of interest to us. We assume that there are no non-perturbative corrections to the anomaly.

The classical piece of $\bar D^2\bar\phi_i{}^j(e^{\mathsf{V}}\phi)_k{}^l$ is $[W'(\phi)_i{}^j-(b\tilde b)_i{}^j]\phi_k{}^l$ and the anomaly is

$$\begin{aligned}&\frac{1}{32\pi^2}(\mathcal{W}_\alpha)_s{}^r(\mathcal{W}^\alpha)_q{}^p(T_r{}^s)_i{}^j{}_m{}^n(T_p{}^q)_n{}^m{}_k{}^l= \\ &\quad=\frac{1}{32\pi^2}\left[(\mathcal{W}_\alpha\mathcal{W}^\alpha)_i{}^l\delta^j_k-(\mathcal{W}_\alpha)_k{}^j(\mathcal{W}^\alpha)_i{}^l-(\mathcal{W}_\alpha)_i{}^l(\mathcal{W}^\alpha)_k{}^j+(\mathcal{W}_\alpha\mathcal{W}^\alpha)_k{}^j\delta^l_i\right].\end{aligned} \tag{2.34}$$

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Similarly, the classical piece of $\bar{D}^2\bar{\tilde{\phi}}_{\tilde{i}}{}^{\tilde{j}}(e^{\mathsf{V}}\tilde{\phi})_{\tilde{k}}{}^{\tilde{l}}$ is $-[\tilde{W}'(\tilde{\phi})_{\tilde{i}}{}^{\tilde{j}} - (\tilde{b}b)_{\tilde{i}}{}^{\tilde{j}}]\tilde{\phi}_{\tilde{k}}{}^{\tilde{l}}$ and the anomaly is

$$\frac{1}{32\pi^2}(\tilde{\mathcal{W}}_\alpha)_{\tilde{s}}{}^{\tilde{r}}(\tilde{\mathcal{W}}^\alpha)_{\tilde{q}}{}^{\tilde{p}}(T_{\tilde{r}}{}^{\tilde{s}})_{\tilde{i}}{}^{\tilde{j}}{}_{\tilde{m}}{}^{\tilde{n}}(T_{\tilde{p}}{}^{\tilde{q}})_{\tilde{n}}{}^{\tilde{m}}{}_{\tilde{k}}{}^{\tilde{l}} = \tag{2.35}$$

$$= \frac{1}{32\pi^2}\left[(\tilde{\mathcal{W}}_\alpha\tilde{\mathcal{W}}^\alpha)_{\tilde{i}}{}^{\tilde{l}}\delta_{\tilde{k}}^{\tilde{j}} - (\tilde{\mathcal{W}}_\alpha)_{\tilde{k}}{}^{\tilde{j}}(\tilde{\mathcal{W}}^\alpha)_{\tilde{i}}{}^{\tilde{l}} - (\tilde{\mathcal{W}}_\alpha)_{\tilde{i}}{}^{\tilde{l}}(\tilde{\mathcal{W}}^\alpha)_{\tilde{k}}{}^{\tilde{j}} + (\tilde{\mathcal{W}}_\alpha\tilde{\mathcal{W}}^\alpha)_{\tilde{k}}{}^{\tilde{j}}\delta_{\tilde{i}}^{\tilde{l}}\right].$$

We will now consider various anomalous currents generalizing the above expressions. These currents all satisfy $0 = \langle \bar{D}^2 J\rangle$ in any supersymmetric vacuum.

Quadratic relations. As a first example we consider the anomaly equation

$$\begin{aligned}
0 &= \frac{1}{32\pi^2}\left\langle \bar{D}^2\mathrm{tr}\left(\bar{\tilde{b}}\, e^{\mathsf{V}}\frac{\mathcal{W}_\alpha}{z-\phi}\, b\, \frac{\tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\right)\right\rangle = \frac{1}{32\pi^2}\left\langle \left(\frac{\mathcal{W}_\alpha}{z-\phi}\right)_j{}^k \left(\frac{\tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\right)_{\tilde{l}}{}^{\tilde{i}} \bar{D}^2\bar{\tilde{b}}_{\tilde{i}}{}^j(e^{\mathsf{V}}b)_k{}^{\tilde{l}}\right\rangle \\
&= \frac{1}{32\pi^2}\left\{\left\langle \mathrm{tr}\left(\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}\right)\mathrm{tr}\left(\frac{\tilde{\mathcal{W}}_\beta\tilde{\mathcal{W}}^\beta}{z-\tilde{\phi}}\right)\right\rangle - \left\langle \mathrm{tr}\left(\frac{\phi}{z-\phi}b\frac{\tilde{\mathcal{W}}_\alpha\tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\tilde{b}\right)\right\rangle + \right.\\
&\qquad\left. + \left\langle \mathrm{tr}\left(\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}b\frac{\tilde{\phi}}{z-\tilde{\phi}}\tilde{b}\right)\right\rangle\right\} \\
&= R(z)\tilde{R}(z) + \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(b\frac{\tilde{\mathcal{W}}_\alpha\tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\tilde{b}\right)\right\rangle - \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\tilde{b}\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}b\right)\right\rangle,
\end{aligned} \tag{2.36}$$

where we introduce

$$R(z) \equiv -\frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}\right)\right\rangle, \qquad \tilde{R}(z) \equiv -\frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{\tilde{\mathcal{W}}_\alpha\tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\right)\right\rangle. \tag{2.37}$$

A few words of explanation are in order. We have dropped covariantization with e^{V} and $e^{-\mathsf{V}}$ since this will not affect the chiral part [3]. We have used (2.33) together with the fact that in the chiral ring no more than two $\mathcal{W}_\alpha$'s and $\tilde{\mathcal{W}}_\alpha$'s can have their gauge indices contracted [3]. In addition we have also used the factorization of the expectation values in the chiral ring [16, 3], and made repeated use of the relations (2.32). Similar considerations will be used throughout this section.

Next we consider (here $f(z)$ is a polynomial; see [3] for similar calculations)

$$\begin{aligned}
0 &= \frac{1}{32\pi^2}\left\langle \bar{D}^2\mathrm{tr}\left(\bar{\phi}\, e^{\mathsf{V}}\frac{f(\phi)\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}\right)\right\rangle \\
&= R(z)^2 f(z) - \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\tilde{b}\frac{f(\phi)\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}b\right)\right\rangle - \frac{1}{(32\pi^2)^2}\sum_m c_m\sum_{k=0}^{m-1}\mathrm{tr}(\phi^k\mathcal{W}_\alpha\mathcal{W}^\alpha)\,\times \\
&\quad \times \mathrm{tr}(\phi^{m-k-1}\mathcal{W}_\beta\mathcal{W}^\beta) + \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{W'(\phi)f(\phi)\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}\right)\right\rangle,
\end{aligned}$$

where we have used (2.34) together with the fact that $\frac{f(z)-f(\phi)}{z-\phi} \equiv \sum_m c_m\phi^m$ is a polynomial.

In particular, for $f(z)=1$ we get

$$R(z)^2 - W'(z)R(z) = \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\tilde{b}\,\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}\,b\right)\right\rangle + \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\phi)}{z-\phi}\mathcal{W}_\alpha\mathcal{W}^\alpha\right)\right\rangle. \tag{2.38}$$

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Analogously, one readily derives

$$0 = \frac{1}{32\pi^2}\left\langle \bar{D}^2 \mathrm{tr}\left(\bar{\tilde{\phi}}\, e^{\mathsf{V}} \frac{f(\tilde{\phi})\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\right)\right\rangle$$

$$= \tilde{R}(z)^2 f(z) + \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(b\frac{f(\tilde{\phi})\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\tilde{b}\right)\right\rangle - \frac{1}{(32\pi^2)^2}\sum_m c_m \sum_{k=0}^{m-1} \mathrm{tr}(\tilde{\phi}^k \tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha)\times$$

$$\times \mathrm{tr}(\tilde{\phi}^{m-k-1}\tilde{\mathcal{W}}_\beta \tilde{\mathcal{W}}^\beta) - \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{\tilde{W}'(\phi) f(\tilde{\phi})\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\right)\right\rangle, \tag{2.39}$$

and

$$\tilde{R}(z)^2 + \tilde{W}'(z)\tilde{R}(z) = -\frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(b\frac{\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\tilde{b}\right)\right\rangle - \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{\tilde{W}'(z)-\tilde{W}'(\tilde{\phi})}{z-\tilde{\phi}}\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha\right)\right\rangle. \tag{2.40}$$

The similarity of eqs. (2.36), (2.38) and (2.40) with eqs. (2.5), (2.7) and (2.10) is obvious. Combining eqs. (2.36)-(2.40) to eliminate b and $\tilde{b}$, we find

$$R(z)^2 + \tilde{R}(z)^2 - R(z)\tilde{R}(z) - W'(z)R(z) + \tilde{W}'(z)\tilde{R}(z) = r_1(z) \tag{2.41}$$

with

$$r_1(z) = \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\phi)}{z-\phi}\mathcal{W}_\alpha \mathcal{W}^\alpha\right)\right\rangle - \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{\tilde{W}'(z)-\tilde{W}'(\tilde{\phi})}{z-\tilde{\phi}}\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha\right)\right\rangle. \tag{2.42}$$

The above two equations are the gauge theory analogs of the matrix-model results (2.12) and (2.13). Since the effect of $r_1(z)$ in (2.41) is to eliminate the positive powers in the Laurent expansion of $-W'(z)R(z)+\tilde{W}'(z)\tilde{R}(z)$, the polynomial $r_1(z)$ has the same function as in the matrix model. The two equations are therefore equivalent and we may identify

$$R(z) = \omega(z)\,, \qquad \tilde{R}(z) = \tilde{\omega}(z)\,. \tag{2.43}$$

By looking at the other equations above one may also identify

$$\begin{aligned} g_{\mathrm{s}}\left\langle \mathrm{tr}\left(B\, f(\tilde{\Phi})\, \tilde{B}\right)\right\rangle &= -\frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(b\, f(\tilde{\phi})\tilde{\mathcal{W}}_\alpha \mathcal{W}^\alpha\, \tilde{b}\right)\right\rangle, \\ g_{\mathrm{s}}\left\langle \mathrm{tr}\left(\tilde{B}\, f(\Phi)\, B\right)\right\rangle &= -\frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\tilde{b}\, f(\phi)\mathcal{W}_\alpha \mathcal{W}^\alpha\, b\right)\right\rangle. \end{aligned} \tag{2.44}$$

Cubic relations. Strictly speaking, equation (2.41) involves both resolvents so we need one more relation before we can make the identifications (2.43). Such a relation is obtained if we can show that the cubic loop equation (2.23), (2.28) is also realized in the gauge theory. Given the close correspondence between the gauge theory and matrix model expressions noted above, the only thing we need to check is that (2.24), (2.25) and (2.26) are also realized in gauge theory consistent with the above identifications. If this is true then the cubic equation will follow in the same way as in the matrix model analysis.

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The gauge-theory analog of (2.24) is

$$
\begin{aligned}
0 &= \frac{1}{32\pi^2}\left\langle \bar{D}^2 \,\mathrm{tr}\left(\bar{\tilde{\phi}}\, e^{V}\, \tilde{b}\, b\, \frac{\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\right)\right\rangle \\
&= -\frac{1}{32\pi^2}\tilde{R}(z)\left\langle \mathrm{tr}\left(b\frac{\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\tilde{b}\right)\right\rangle - \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(b\frac{\tilde{W}'(\tilde{\phi})}{z-\tilde{\phi}}\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha \tilde{b}\right)\right\rangle + \\
&\quad + \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\tilde{b}b\frac{\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\tilde{b}b\right)\right\rangle. \qquad (2.45)
\end{aligned}
$$

Similarly, consideration of $\frac{1}{32\pi^2}\langle \bar{D}^2\,\mathrm{tr}\Big(\bar{\phi}\, e^{V}\, b\, \tilde{b}\, \frac{\mathcal{W}_\alpha \mathcal{W}^\alpha}{z-\phi}\Big)\rangle$ leads to the analog of (2.25):

$$
\begin{aligned}
0 &= -\frac{1}{32\pi^2}R(z)\left\langle \mathrm{tr}\left(\tilde{b}\frac{\mathcal{W}_\alpha \mathcal{W}^\alpha}{z-\phi}b\right)\right\rangle + \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\tilde{b}\frac{W'(\phi)}{z-\phi}\mathcal{W}_\alpha \mathcal{W}^\alpha b\right)\right\rangle - \\
&\quad -\frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(b\tilde{b}\frac{\mathcal{W}_\alpha \mathcal{W}^\alpha}{z-\phi}b\tilde{b}\right)\right\rangle. \qquad (2.46)
\end{aligned}
$$

Finally, the analog of (2.26) is

$$
\begin{aligned}
0 &= \frac{1}{32\pi^2}\left\langle \bar{D}^2\,\mathrm{tr}\left(\bar{b}\, e^{V}\frac{\tilde{\mathcal{W}}_\alpha}{z-\tilde{\phi}}\,\tilde{b}\, b\, \tilde{b}\, \frac{\mathcal{W}^\alpha}{z-\phi}\right)\right\rangle \\
&= -\frac{1}{32\pi^2}\tilde{R}(z)\left\langle \mathrm{tr}\left(\tilde{b}\frac{\mathcal{W}_\alpha \mathcal{W}^\alpha}{z-\phi}b\right)\right\rangle - \frac{1}{32\pi^2}R(z)\left\langle \mathrm{tr}\left(b\frac{\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\tilde{b}\right)\right\rangle + \\
&\quad + \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\tilde{b}\, b\frac{\tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha}{z-\tilde{\phi}}\tilde{b}\, b\right)\right\rangle - \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(b\,\tilde{b}\frac{\mathcal{W}_\alpha \mathcal{W}^\alpha}{z-\phi}b\,\tilde{b}\right)\right\rangle. \qquad (2.47)
\end{aligned}
$$

This completes the discussion of the equivalence of the matrix-model loop equations and gauge-theory anomaly equations.

It is worth noting that we did not have to use the entire chiral ring (which includes expressions with arbitrary many bifundamental fields) to derive the equations which determine $R(z)$, $\tilde{R}(z)$.

Relation between gauge-theory and matrix-model expectation values. It is also of obvious interest to look for equations which determine

$$
T(z) \equiv \left\langle \mathrm{tr}\left(\frac{1}{z-\phi}\right)\right\rangle, \qquad \tilde{T}(z) \equiv \left\langle \mathrm{tr}\left(\frac{1}{z-\tilde{\phi}}\right)\right\rangle, \qquad (2.48)
$$

since these expressions act as generating functions for the gauge-theory expectation values $\mathrm{tr}(\phi^k)$ and $\mathrm{tr}(\tilde{\phi}^k)$, whereas $R(z)$ and $\tilde{R}(z)$ (which by the above analysis are equal to $\omega(z)$ and $\tilde{\omega}(z)$, respectively) are the generating functions (2.3) for the matrix-model expectation values $\mathrm{tr}(\Phi^k)$ and $\mathrm{tr}(\tilde{\Phi}^k)$.

Before discussing the $\mathrm{U}(N)\times\mathrm{U}(N)$ case, let us recall the case of the $\mathrm{U}(N)$ theory with adjoint matter only. In a supersymmetric vacuum the equations governing this model

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are [3]

$$R(z)^2 - W'(z)R(z) = \frac{1}{4}f(z) = \frac{1}{32\pi^2}\Big\langle \mathrm{tr}\Big(\frac{W'(z)-W'(\phi)}{z-\phi}\mathcal{W}_\alpha\mathcal{W}^\alpha\Big)\Big\rangle, \quad (2.49)$$

$$2R(z)T(z) - W'(z)T(z) = \frac{1}{4}c(z) = -\Big\langle \mathrm{tr}\Big(\frac{W'(z)-W'(\phi)}{z-\phi}\Big)\Big\rangle. \quad (2.50)$$

Recalling the definition of the glueball field, $S = -\frac{1}{32\pi^2}\mathrm{tr}(\mathcal{W}_\alpha\mathcal{W}^\alpha)$, we see that the second equation is formally the derivative of the first equation, with the identifications $T(z) = \frac{\partial}{\partial S}R(z)$ and $c(z) = \frac{\partial}{\partial S}f(z)$. On the gauge theory side, this of course does not quite make sense; on the matrix model side, however, where S is just a parameter ($= g_{\mathrm{s}}M$), it makes sense to take a derivative with respect to S. Since $R(z)$ in the gauge theory is identified with $\omega(z)$ in the matrix model, we are therefore led to the equation[5]

$$T(z) = \frac{\partial}{\partial S}\omega\,. \quad (2.51)$$

Precisely this formula was proposed in refs. [17, 18] (taking into account differences in conventions and recalling that ω in the formula above is only the leading term in the genus expansion of the resolvent, ω_0).

A similar analysis can be carried out in the $\mathrm{U}(N)$ model with additional matter in the fundamental representation and a superpotential of the form $\mathrm{tr}[W(\phi)]+\sum_{I=1}^{N_f}\tilde{q}^I(\phi+m_I)q^I$. For this particular case we have, using the results in [5]

$$\begin{aligned} R(z)^2 - W'(z)R(z) &= \frac{1}{4}f(z) = \frac{1}{32\pi^2}\Big\langle \mathrm{tr}\Big(\frac{W'(z)-W'(\phi)}{z-\phi}\mathcal{W}_\alpha\mathcal{W}^\alpha\Big)\Big\rangle, \\ 2R(z)T(z) - W'(z)T(z) - \sum_{I=1}^{N_f}\tilde{q}^I\frac{1}{z-\phi}q^I &= \frac{1}{4}c(z) = -\Big\langle \mathrm{tr}\Big(\frac{W'(z)-W'(\phi)}{z-\phi}\Big)\Big\rangle. \end{aligned} \quad (2.52)$$

The first equation is the same as in the case without fundamental matter (2.50), but the second equation has an extra contribution. We also have [5]

$$\tilde{q}^I\frac{(\phi+m_I)}{z-\phi}q^I = R(z) \qquad (\text{no sum over } I)\,. \quad (2.53)$$

It follows from this equation that

$$\frac{R(z)}{z+m_I} = \tilde{q}^I\frac{1}{z-\phi}q^I - \tilde{q}^I\frac{1}{z+m_I}q^I = \tilde{q}^I\frac{1}{z-\phi}q^I + \frac{R(-m_I)}{z+m_I}\,. \quad (2.54)$$

Using this result to eliminate the q-dependence in (2.52) and using the argumentation above we are led to the identification (where $R(z) = \omega_0(z)$)

$$T(z) = \frac{\partial}{\partial S}\omega_0(z) + \frac{1}{2\omega_0(z)-W'(z)}\sum_I\frac{\omega_0(z)-\omega_0(-m_I)}{z+m_I} = \frac{\partial}{\partial S}\omega_0(z) + \omega_{1/2}(z)\,, \quad (2.55)$$

[5]Here, and in subsequent equations, $\partial/\partial S$ should be identified with $\sum_i N_i\partial/\partial S_i$, as can be seen by considering the $z\to\infty$ part of this equation. When we compare with the results in ref. [18], we set $N_i = 1$ for all i.

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where $\omega_{1/2}(z)$ is the subleading (disk) contribution [18] in the topological expansion of the resolvent: $\omega(z) = \omega_0(z) + g_{\rm s}\omega_{1/2}(z) + \cdots$. Note that the expression (2.55) precisely agrees with eqs. (8.7), (8.9) in (version 3 of) ref. [18] (after taking into account differences in conventions: $\omega_0 = -S\omega_{\rm s}$, $\omega_{1/2} = -\omega_{\rm d}$). It is interesting to note that on the matrix-model side the extra term in (2.55) compared to (2.51) comes from a subleading (disk) effect in the matrix-model loop equation [18], whereas on the gauge theory side it arises from another equation, rather than from a subleading term. We also note that (2.55) and the more explicit expression derived from it (in [18, eq. (8.15)], valid when $N_f < N$) agrees with (a special case of) the expression for $T(z)$ given in the very recent paper [7] (cf. eqs (3.10), (3.11) of that paper), using in particular the result $\langle \mathrm{tr}(\phi^k)\rangle = \langle \mathrm{tr}(\phi^k)\rangle_{\rm classical}$ for $k \leq N$.

Let us now return to the $\mathrm{U}(N)\times\mathrm{U}(N)$ model. By repeating the steps which lead to (2.41) using analogous currents, but without the $\mathcal{W}_\alpha\mathcal{W}^\alpha$ and $\tilde{\mathcal{W}}_\alpha\tilde{\mathcal{W}}^\alpha$ factors, one may derive

$$\begin{aligned} 0 &= -\bar{D}^2\left\langle \mathrm{tr}\left(\bar{\phi}\, e^{\mathsf{V}} \frac{1}{z-\phi}\right)\right\rangle - \bar{D}^2\left\langle \mathrm{tr}\left(\bar{\tilde{\phi}}\, e^{\mathsf{V}} \frac{1}{z-\tilde{\phi}}\right)\right\rangle + \bar{D}^2\left\langle \mathrm{tr}\left(\bar{\tilde{b}}\, e^{\mathsf{V}} \frac{1}{z-\phi}\, b\, \frac{1}{z-\tilde{\phi}}\right)\right\rangle \\ &= 2R(z)T(z) + 2\tilde{R}(z)\tilde{T}(z) - R(z)\tilde{T}(z) - \tilde{R}(z)T(z) - W'(z)T(z) + \tilde{W}'(z)\tilde{T}(z) - c_1(z) \end{aligned} \tag{2.56}$$

(where we have dropped terms involving — after factorization — $\langle \mathrm{tr}(\frac{\mathcal{W}_\alpha}{z-\phi})\rangle$ and $\langle \mathrm{tr}(\frac{\tilde{\mathcal{W}}_\alpha}{z-\tilde{\phi}})\rangle$ since such terms vanish in a supersymmetric vacuum) and the polynomial $c_1(z)$ is explicitly given by

$$c_1(z) = -\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\phi)}{z-\phi}\right)\right\rangle + \left\langle \mathrm{tr}\left(\frac{\tilde{W}'(z)-\tilde{W}'(\tilde{\phi})}{z-\tilde{\phi}}\right)\right\rangle. \tag{2.57}$$

We note that (2.56) can formally be obtained by taking derivatives of (2.41). We are therefore led to suggest the identifications[6]

$$T(z) = \left[\frac{\partial}{\partial S} + \frac{\partial}{\partial \tilde{S}}\right]\omega\,, \qquad \tilde{T}(z) = \left[\frac{\partial}{\partial S} + \frac{\partial}{\partial \tilde{S}}\right]\tilde{\omega}\,. \tag{2.58}$$

which fit nicely into the structure given by the results (2.51), (2.55). Since we have two unknowns but only one equation, we can not argue unambiguously in favor of the above identifications, but consideration of the cubic equation for $T(z)$, $\tilde{T}(z)$ analogous to the one for $R(z)$, $\tilde{R}(z)$ presumably also leads to (2.58), although we have not checked this explicitly.

3. $\mathrm{U}(N)$ with symmetric or antisymmetric matter: I

In this section we consider the $\mathcal{N}=1$ $\mathrm{U}(N)$ supersymmetric gauge theory with one chiral superfield $\phi_i{}^j$ transforming in the adjoint representation of the gauge group, one chiral superfield x_{ij} transforming in either the symmetric ($\square\!\square$) or the antisymmetric ($\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$) representation, and one chiral superfield $\tilde{x}^{ij}$ transforming in the conjugate representation. We treat the cases of the symmetric and antisymmetric representations simultaneously

[6] This result may also be obtained via the method in refs. [22, 17].

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by assuming that x, $\tilde{x}$ satisfy $x^T = \beta x$ and $\tilde{x}^T = \beta \tilde{x}$, where $\beta = 1$ for the symmetric representation, and $\beta = -1$ for the antisymmetric representation.

The superpotential of the gauge theory is taken to be of the form[7]

$$\mathcal{W}(\phi, x, \tilde{x}) = \mathrm{tr}[W(\phi) - \tilde{x}\,\phi\,x]\,, \tag{3.1}$$

where $W(\phi) = \sum_{m=1}^{N+1}(g_m/m)\mathrm{tr}(\phi^m)$. This superpotential can be viewed as a deformation of an $\mathcal{N} = 2$ theory.

Below, after deriving the loop equations of the matrix model (including the first subleading contribution in the $1/M$ expansion) we establish the non-perturbative equivalence of the holomorphic sector of the above gauge theory to the associated matrix model, by showing how the matrix-model loop equations are encoded in the gauge theory.

The extension of the perturbative argument given in ref. [2] to include the models considered in this section will be treated in section 4.

3.1 Matrix model analysis

The partition function for the (holomorphic) matrix model is taken to be[8]

$$Z = \int \mathrm{d}\Phi\,\mathrm{d}X\,\mathrm{d}\tilde{X}\,\mathrm{e}^{-\frac{1}{g_s}\mathrm{tr}[W(\Phi) - \tilde{X}\Phi X]}\,, \tag{3.2}$$

where $X^T = \beta X$, $\tilde{X}^T = \beta\tilde{X}$, and $\beta = 1$ (-1) for $\Box\!\Box$ $\left(\begin{smallmatrix}\Box\\\Box\end{smallmatrix}\right)$.

We are interested in the planar limit of the matrix model, i.e. the limit in which $g_s \to 0$ and $M \to \infty$, keeping $S = g_s M$ fixed. The above matrix model is closely related to the $O(n)$ matrix model [19] with $n = 1$. The planar saddle-point solution of that model was derived in refs. [20]; see also the recent paper [12] where the planar solution to (3.2) was discussed. In the saddle-point approach, one diagonalizes the matrix Φ and derives equations satisfied by the resolvent[9]

$$\omega(z) = g_s \left\langle \mathrm{tr}\left(\frac{1}{z - \Phi}\right)\right\rangle = g_s \sum_i \frac{1}{z - \lambda_i}\,, \tag{3.3}$$

where matrix-model expectation values are defined via

$$\langle \mathcal{O}(\Phi, X, \tilde{X})\rangle = \frac{1}{Z}\int \mathrm{d}\Phi\,\mathrm{d}X\,\mathrm{d}\tilde{X}\,\mathcal{O}(\Phi, X, \tilde{X})\,\mathrm{e}^{-\frac{1}{g_s}\mathrm{tr}[W(\Phi) - \tilde{X}\Phi X]}\,. \tag{3.4}$$

Some details of the saddle-point analysis are given in appendix A.

Below we derive the equations satisfied by the resolvents using an approach [3] (see also the approach in e.g. [19, 20]) that is close in spirit to the gauge theory analysis given

[7] We do not explicitly include a mass term for the x, $\tilde{x}$ fields although we think of these fields as being massive, cf. footnote 2.

[8] As in the previous section, we use capital letters to denote matrix model quantities. All matrix indices run over M values.

[9] We use an unconventional normalization of the resolvent in order to make the comparison with gauge theory more transparent. Also, in order not to clutter the formulæ we drop the $\langle \cdots \rangle$ when writing expressions in terms of eigenvalues.

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in section 3.2. The discussion closely parallels the one in section 2 (which is not surprising since the models in this section are orientifolds[10] of the one in section 2). (We stress that this method does not assume that the matrices are hermitean.)

Quadratic relations. We begin by considering the Ward identity

$$
\begin{aligned}
0 &= \frac{2g_s^2}{Z}\int \mathrm{d}\Phi\,\mathrm{d}X\,\mathrm{d}\tilde{X}\,\frac{\mathrm{d}}{\mathrm{d}X_{ij}}\left\{\left[\frac{1}{z-\Phi}X\left(\frac{1}{z+\Phi}\right)^T\right]_{ij}\mathrm{e}^{-\frac{1}{g_s}\mathrm{tr}[W(\Phi)-\tilde{X}\Phi X]}\right\}\\
&= g_s^2\left\langle \mathrm{tr}\left(\frac{1}{z-\Phi}\right)\mathrm{tr}\left(\frac{1}{z+\Phi}\right)\right\rangle + \beta g_s^2\left\langle \mathrm{tr}\left(\frac{1}{z-\Phi}\frac{1}{z+\Phi}\right)\right\rangle +\\
&\quad + g_s\left\langle \mathrm{tr}\left[\frac{1}{z-\Phi}X\left(\frac{1}{z+\Phi}\right)^T\tilde{X}\Phi\right]\right\rangle + g_s\left\langle \mathrm{tr}\left[\frac{1}{z+\Phi}X\left(\frac{1}{z-\Phi}\right)^T\tilde{X}\Phi\right]\right\rangle\\
&= -\omega(z)\omega(-z) + \frac{\beta g_s}{2z}[\omega(z){-}\omega(-z)] + g_s\left\langle \mathrm{tr}\left(\tilde{X}\frac{1}{z{-}\Phi}X\right)\right\rangle - g_s\left\langle \mathrm{tr}\left(\tilde{X}\frac{1}{z{+}\Phi}X\right)\right\rangle, \quad (3.5)
\end{aligned}
$$

where we have used factorization of the expectation values in the planar limit. The corrections to factorization go like $\frac{1}{M^2}$ (or g_s^2). In the above expression, we have neglected the $\frac{1}{M^2}$ corrections, but have kept the $\frac{1}{M}$ (or g_s) subleading terms. Note that it is only the terms that are proportional to β that are subleading (this feature is true in all equations in this section); the g_s-dependence in the last two terms in (3.5) is related to our normalizations of X, $\tilde{X}$ and $\omega(z)$, and does not mean that these terms are subleading.

In complete analogy with (2.7) and (2.8) one may derive

$$
\omega(z)^2 - W'(z)\,\omega(z) = -g_s\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\right)\right\rangle - g_s\left\langle \mathrm{tr}\left(\tilde{X}\frac{1}{z-\Phi}X\right)\right\rangle, \quad (3.6)
$$

as well as

$$
\begin{aligned}
g_s\left\langle \mathrm{tr}\left(\tilde{X}\frac{f(z)-f(\Phi)}{z-\Phi}X\right)\right\rangle &= -g_s^2\left\langle \mathrm{tr}\left[\frac{\mathrm{d}}{\mathrm{d}\Phi}\left(\frac{f(z)-f(\Phi)}{z-\Phi}\right)\right]\right\rangle +\\
&\quad + g_s\left\langle \mathrm{tr}\left(\frac{f(z)-f(\Phi)}{z-\Phi}W'(\Phi)\right)\right\rangle. \quad (3.7)
\end{aligned}
$$

Combining (3.5) with (3.6) to eliminate the X-dependent terms, one finds

$$
\omega(z)^2+\omega(-z)^2+\omega(z)\omega(-z)-W'(z)\omega(z)-W'(-z)\omega(-z) = r_1(z)+\frac{\beta g_s}{2z}[\omega(z)-\omega(-z)]\,, \quad (3.8)
$$

where

$$
\begin{aligned}
r_1(z) &= -g_s\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\right)\right\rangle - g_s\left\langle \mathrm{tr}\left(\frac{W'(-z)-W'(\Phi)}{-z-\Phi}\right)\right\rangle\\
&= -g_s\sum_i\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i} - g_s\sum_i\frac{W'(-z)-W'(\lambda_i)}{-z-\lambda_i} \quad (3.9)
\end{aligned}
$$

is a (manifestly even) polynomial of degree at most $N-1$.

[10]Most of the equations in this section can be related to the ones in section 2 by implementing an orientifold projection on the fields. Note, however, that the subleading terms to be discussed below can *not* be obtained this way.

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The β-dependent terms on the r.h.s. of eq. (3.8) are subleading in $g_{\rm s}$ compared to the rest of the terms. We may expand the resolvent in powers of $g_{\rm s}$, i.e. in a topological expansion [21] as $\omega(z) = \sum_{\chi\le 2} g_{\rm s}^{2-\chi}\omega_{1-\chi/2}(z) = \omega_0(z) + g_{\rm s}\,\omega_{1/2}(z) + \cdots$. Here χ is the Euler characteristic, the leading term is the sphere ($\chi = 2$) contribution, and the next term is an $\mathbb{RP}^2$ ($\chi = 1$) contribution. Using this expansion to solve (3.8) order-by-order we find (in agreement with [20])

$$\omega_0(z)^2 + \omega_0(-z)^2 + \omega_0(z)\omega_0(-z) - W'(z)\omega_0(z) - W'(-z)\omega_0(-z) = r_1(z)\,, \tag{3.10}$$

and

$$\begin{aligned}2\,\omega_0(z)\,\omega_{1/2}(z) + 2\,\omega_0(-z)\,\omega_{1/2}(-z) + \omega_0(z)\,\omega_{1/2}(-z) + \omega_0(-z)\,\omega_{1/2}(z)-\\ -W'(z)\,\omega_{1/2}(z) - W'(-z)\,\omega_{1/2}(-z) - \frac{\beta}{2z}[\omega_0(z) - \omega_0(-z)] = 0\,.\end{aligned} \tag{3.11}$$

The cubic algebraic curve. As we now discuss (see also refs. [20, 12]) there is a cubic algebraic curve underlying the model. The linear term in eq. (3.10) can be eliminated by defining

$$\omega_0(z) = u_1(z) + \omega_r(z), \qquad \omega_0(-z) = u_3(z) + \omega_r(-z)\,, \tag{3.12}$$

with

$$\omega_r(z) = \frac{2}{3}W'(z) - \frac{1}{3}W'(-z)\,, \tag{3.13}$$

giving

$$u_1(z)^2 + u_3(z)^2 + u_1(z)u_3(z) = r_0(z) + r_1(z)\,, \tag{3.14}$$

with

$$\begin{aligned}r_0(z) &= \omega_r^2(z) + \omega_r^2(-z) + \omega_r(z)\omega_r(-z)\\ &= \frac{1}{3}\left[W'^2(z) + W'^2(-z) - W'(z)W'(-z)\right],\end{aligned} \tag{3.15}$$

a polynomial of degree $2N$. Multiplying eq. (3.14) by $u_1(z) - u_3(z)$, we find [20]

$$u_1(z)^3 - r(z)u_1(z) = u_3(z)^3 - r(z)u_3(z) \equiv s(z)\,, \tag{3.16}$$

so that $u_1(z)$ and $u_3(z)$ are both roots of the cubic equation

$$0 = u^3 - r(z)\,u - s(z) = (u - u_1(z))(u - u_2(z))(u - u_3(z))\,. \tag{3.17}$$

The absence of the quadratic term show that the third root is $u_2(z) = -u_1(z) - u_3(z)$, and

$$s(z) = u_1(z)u_2(z)u_3(z) = [\omega_0(z) - \omega_r(z)][-\omega_0(z) - \omega_0(-z) + \omega_r(z) + \omega_r(-z)][\omega_0(-z) - \omega_r(-z)]\,, \tag{3.18}$$

which we will show to be a (manifestly even) polynomial below.

Defining $s(z) = s_0(z) + s_1(z)$ with

$$\begin{aligned}s_0(z) &= \omega_r(z)\,\omega_r(-z)[\omega_r(z) + \omega_r(-z)]\\ &= \frac{1}{27}[-W'(z) + 2W'(-z)][2W'(z) - W'(-z)][W'(z) + W'(-z)]\,,\end{aligned} \tag{3.19}$$

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a polynomial of degree $3N$, we can rewrite the cubic equation as

$$\begin{aligned} r_1(z)\,u + s_1(z) &= u^3 - r_0(z)u - s_0(z) \\ &= (u+\omega_r(z))(u-\omega_r(z)-\omega_r(-z))(u+\omega_r(-z))\,. \end{aligned} \tag{3.20}$$

From eqs. (3.18) and (3.19) it follows that

$$\begin{aligned} s_1(z) \;=\; &-\omega_0(z)\,\omega_0(-z)[\omega_0(z)+\omega_0(-z)] + \frac{2}{3}[W'(z)+W'(-z)]\,\omega_0(z)\,\omega_0(-z) + \\ &+\omega_r(-z)\,[\omega_0(z)^2 - W'(z)\omega_0(z)] + \omega_r(z)\,[\omega_0(-z)^2 - W'(-z)\omega_0(-z)]\,. \end{aligned} \tag{3.21}$$

We will show below that $s_1(z)$ is an even polynomial of degree at most $2N-1$.

Cubic relations. Using Ward identities, we now show how to obtain the relation (3.21) with an explicit expression for $s_1(z)$. In complete analogy with eq. (2.24) and (2.25) we have

$$0 = g_{\mathrm{s}}\,\omega(z)\left\langle \mathrm{tr}\left(\tilde{X}\frac{1}{z-\Phi}X\right)\right\rangle - g_{\mathrm{s}}\left\langle \mathrm{tr}\left(\tilde{X}\frac{W'(\Phi)}{z-\Phi}X\right)\right\rangle + g_{\mathrm{s}}\left\langle \mathrm{tr}\left(X\tilde{X}\frac{1}{z-\Phi}\tilde{X}X\right)\right\rangle. \tag{3.22}$$

It can also be shown that

$$\begin{aligned} 0 \;&=\; 2\frac{g_{\mathrm{s}}^2}{Z}\int \mathrm{d}\Phi\,\mathrm{d}X\,\mathrm{d}\tilde{X}\,\frac{\mathrm{d}}{\mathrm{d}X_{ij}}\left\{\left[\frac{1}{z-\Phi}X\tilde{X}X\left(\frac{1}{z+\Phi}\right)^T\right]_{ij}\mathrm{e}^{-\frac{1}{g_s}\mathrm{tr}[W(\Phi)-\tilde{X}\Phi X]}\right\} \\ &= g_{\mathrm{s}}\,\omega(z)\left\langle \mathrm{tr}\left(\tilde{X}\frac{1}{z+\Phi}X\right)\right\rangle - g_{\mathrm{s}}\,\omega(-z)\left\langle \mathrm{tr}\left(\tilde{X}\frac{1}{z-\Phi}X\right)\right\rangle + g_{\mathrm{s}}\left\langle \mathrm{tr}\left(X\tilde{X}\frac{1}{z-\Phi}X\tilde{X}\right)\right\rangle - \\ &\quad - g_{\mathrm{s}}\left\langle \mathrm{tr}\left(X\tilde{X}\frac{1}{z+\Phi}X\tilde{X}\right)\right\rangle + \frac{\beta g_{\mathrm{s}}^2}{z}\left[\left\langle \mathrm{tr}\left(\tilde{X}\frac{1}{z-\Phi}X\right)\right\rangle + \left\langle \mathrm{tr}\left(\tilde{X}\frac{1}{z+\Phi}X\right)\right\rangle\right]. \end{aligned} \tag{3.23}$$

Combining eqs. (3.22) and (3.23), and using eqs. (3.5) and (3.6), we find

$$\begin{aligned} -\omega(z)\omega(-z)\,[\omega(z)+\omega(-z)] \;=\; &-g_{\mathrm{s}}\left\langle \mathrm{tr}\left(\tilde{X}\frac{W'(\Phi)}{z-\Phi}X\right)\right\rangle + g_{\mathrm{s}}\left\langle \mathrm{tr}\left(X\frac{W'(\Phi)}{z+\Phi}\tilde{X}\right)\right\rangle + \\ &+\frac{\beta g_{\mathrm{s}}}{2z}\left[\omega(z)^2 - \omega(-z)^2\right] - \\ &-\frac{\beta g_{\mathrm{s}}^2}{z}\left\langle \mathrm{tr}\left(\frac{W'(\Phi)}{z-\Phi}\right) + \mathrm{tr}\left(\frac{W'(\Phi)}{z+\Phi}\right)\right\rangle. \end{aligned}$$

Only the first two (non β-dependent) terms on the r.h.s. of this equation contribute to the leading-order (sphere) piece on the l.h.s., i.e. to $-\omega_0(z)\omega_0(-z)\,[\omega_0(z)+\omega_0(-z)]$. Considering only the leading terms in eq. (3.24) and using eqs. (3.5)–(3.7), one obtains (by comparison with (3.21)) an explicit expression for $s_1(z)$ in terms of the matrix model vevs $\langle \mathrm{tr}(\Phi^k)\rangle$

$$\begin{aligned} s_1(z) \;=\; &-g_{\mathrm{s}}\,\omega_r(-z)\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\right)\right\rangle - g_{\mathrm{s}}^2\left\langle \mathrm{tr}\left[\frac{\mathrm{d}}{\mathrm{d}\Phi}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}\right)\right]\right\rangle + \\ &+g_{\mathrm{s}}\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\Phi)}{z-\Phi}W'(\Phi)\right)\right\rangle + (z\to -z)\,. \end{aligned} \tag{3.24}$$

Using eq. (2.29), we can rewrite this more explicitly, in terms of the eigenvalues λ_i of Φ:

$$
\begin{aligned}
s_1(z) \;=\; & -g_\mathrm{s}\,\omega_r(-z)\sum_i\left[\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i}\right] - 2g_\mathrm{s}^2\sum_{i\neq j}\frac{1}{\lambda_i-\lambda_j}\left[\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i}\right] + \\
& + g_\mathrm{s}\sum_i W'(\lambda_i)\left[\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i}\right] + (z\to -z)\,.
\end{aligned}
\tag{3.25}
$$

Finally, using the saddle point equations (A.7), we may rewrite this as

$$
s_1(z) = -g_\mathrm{s}\,\omega_r(-z)\sum_i\left[\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i}\right] - g_\mathrm{s}^2\sum_{i,j}\frac{1}{\lambda_i+\lambda_j}\left[\frac{W'(z)-W'(\lambda_i)}{z-\lambda_i}\right] + (z\to -z)\,.
\tag{3.26}
$$

From this expression it is clear that $s_1(z)$ is an even polynomial of degree at most $2N-1$. The result (3.26) also appeared recently in ref. [12], although using a different method of derivation. Next we turn to the gauge theory analysis.

3.2 Gauge theory analysis

Below we show that the matrix-model loop equations discussed above can be obtained in the gauge theory from generalizations of the Konishi anomaly equations.

As in section 2, it is sufficient to study the chiral part of the anomaly equations. In the chiral ring we have $0 = [\bar{Q}^{\dot\alpha}, D_{\alpha\dot\alpha}F] = \mathcal{W}_\alpha F$, where $\mathcal{W}_\alpha$ is the gauge spinor superfield; more explicitly, $\mathcal{W}_\alpha = \mathcal{W}_\alpha^A T^A$, where T^A are the representation matrices appropriate for the action of the gauge field on the field F. Using the explicit expressions for T^A given in appendix B one obtains the identities (valid in the chiral ring)

$$
[\mathcal{W}_\alpha,\phi] = 0\,,\qquad \mathcal{W}_\alpha x = -x(\mathcal{W}_\alpha)^T\,,\qquad \tilde{x}\mathcal{W}_\alpha = -(\mathcal{W}_\alpha)^T\tilde{x}\,,
\tag{3.27}
$$

which will be repeatedly used below.

The basic building blocks that we will need are the (anomalous) currents $\bar{\tilde{x}}^{ij}(e^V x)_{kl}$ and $\bar\phi_i{}^j(e^V\phi)_k{}^l$. Using the superpotential (3.1), one finds the classical piece of $\bar{D}^2\bar{\tilde{x}}^{ij}(e^V x)_{kl}$ to be $-(\tilde{x}\phi)^{[ij)}x_{kl} = -\frac{1}{2}[\tilde{x}\phi+\phi^T\tilde{x}]^{ij}x_{kl}$; the anomalous contribution [4] is

$$
\frac{1}{32\pi^2}(\mathcal{W}_\alpha)_n{}^m(\mathcal{W}^\alpha)_q{}^p(T_m{}^n)^{ij}_{rs}(T_p{}^q)^{sr}_{kl} = \frac{1}{32\pi^2}\left[2(\mathcal{W}_\alpha\mathcal{W}^\alpha)_{[k}{}^{[j}\delta^{i)}_{l)} + 2(\mathcal{W}_\alpha)_{[k}{}^{[j}(\mathcal{W}^\alpha)_{l)}{}^{i)}\right].
\tag{3.28}
$$

(The $[\;)$ notation is explained in appendix B.) The classical piece of $\bar{D}^2\bar\phi_i{}^j(e^V\phi)_k{}^l$ is given by $W'(\phi)_i{}^j\phi_k{}^l - (x\tilde{x})_i{}^j\phi_k{}^l$ and the anomaly is the same as in eq. (2.34).

As in section 2, we now generalize these currents.[11] The approach is very similar to the one in section 2 so we suppress the details.

[11] One can argue [3] that there should be no chiral, perturbative corrections to the anomalies of the currents, but what about non-perturbative corrections? The SU($2N$) + ⊟ theory does have composite pfaffian operators which might affect the discussion. However, since we are dealing with U(N) it seems that such operators should not be present.

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Quadratic relation. It can be shown that

$$0 = \frac{1}{32\pi^2}\left\langle \bar{D}^2\left[\mathrm{tr}\left(\bar{\phi}\, e^V \frac{\mathcal{W}_\alpha \mathcal{W}^\alpha}{z-\phi}\right) + \mathrm{tr}\left(\bar{\phi}\, e^V \frac{\mathcal{W}_\alpha \mathcal{W}^\alpha}{-z-\phi}\right) + 2\,\mathrm{tr}\left(\bar{\tilde{x}}\, e^V \frac{\mathcal{W}_\alpha}{z-\phi} x \left(\frac{\mathcal{W}^\alpha}{z+\phi}\right)^T\right)\right]\right\rangle$$
$$= R(z)^2 + R(-z)^2 + R(z)R(-z) - W'(z)R(z) - W'(-z)R(-z) - r_1(z)\,, \tag{3.29}$$

with

$$r_1(z) = \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{W'(z)-W'(\phi)}{z-\phi}\mathcal{W}_\alpha\mathcal{W}^\alpha\right)\right\rangle + \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\frac{W'(-z)-W'(\phi)}{-z-\phi}\mathcal{W}_\alpha\mathcal{W}^\alpha\right)\right\rangle, \tag{3.30}$$

where we have used (here and throughout) the fact that in the chiral ring no more than two $\mathcal{W}_\alpha$'s can have their gauge indices contracted [3], together with the factorization property [16, 3] and also the relations (3.27), valid in the chiral ring.

Cubic relation. One may also derive (3.21) in the gauge theory. To show this it is sufficient to obtain the gauge theory analogues of (3.22) and (3.23). This is done by considering

$$\begin{aligned} 0 &= \frac{1}{32\pi^2}\left\langle \bar{D}^2\,\mathrm{tr}\left(\bar{\phi}\, e^V x\,\tilde{x}\,\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}\right)\right\rangle \\ &= -\frac{1}{32\pi^2}R(z)\left\langle \mathrm{tr}\left(\tilde{x}\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}x\right)\right\rangle + \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(\tilde{x}\frac{W'(\phi)}{z-\phi}\mathcal{W}_\alpha\mathcal{W}^\alpha x\right)\right\rangle - \\ &\quad - \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(x\tilde{x}\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}x\tilde{x}\right)\right\rangle, \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} 0 &= -\frac{1}{32\pi^2}\left\langle \bar{D}^2\,\mathrm{tr}\left(\bar{\tilde{x}}\, e^V \frac{\mathcal{W}_\alpha}{z-\phi}\, x\,\tilde{x}\,x\,\left(\frac{\mathcal{W}^\alpha}{z+\phi}\right)^T\right)\right\rangle \\ &= -\frac{1}{32\pi^2}R(z)\left\langle \mathrm{tr}\left(\tilde{x}\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z+\phi}x\right)\right\rangle + \frac{1}{32\pi^2}R(-z)\left\langle \mathrm{tr}\left(\tilde{x}\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}x\right)\right\rangle - \\ &\quad - \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(x\,\tilde{x}\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\phi}x\,\tilde{x}\right)\right\rangle + \frac{1}{32\pi^2}\left\langle \mathrm{tr}\left(x\,\tilde{x}\frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z+\phi}x\,\tilde{x}\right)\right\rangle. \end{aligned}$$

By comparison of the leading (β-independent) parts of the matrix-model expressions with the above gauge theory equations, we find that they agree provided we identify

$$R(z) = \omega_0(z)\,. \tag{3.32}$$

We note that it was not necessary to consider the entire chiral ring (i.e. operators with arbitrary many symmetric or antisymmetric fields) to obtain this result. It is also possible to derive relations between expectation values involving the symmetric (or antisymmetric) fields, e.g.

$$-\frac{1}{32\pi^2}\left\langle \mathrm{tr}(\tilde{x}\, f(\phi)\mathcal{W}_\alpha\mathcal{W}^\alpha\, x)\right\rangle = g_{\mathrm{s}}\left\langle \mathrm{tr}(\tilde{X}\, f(\Phi)\, X)\right\rangle. \tag{3.33}$$

Notice that no subleading terms appeared in the gauge theory equations. The role of the subleading terms in the matrix model expressions will become clear below.

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Relation between gauge-theory and matrix-model expectation values. The generating function for the gauge theory expectation values $\langle \mathrm{tr}(\phi^k)\rangle$ is

$$T(z) \equiv \left\langle \mathrm{tr}\left(\frac{1}{z-\phi}\right)\right\rangle. \tag{3.34}$$

An equation involving this function, analogous to (3.29), can be derived by dropping the $\mathcal{W}_\alpha \mathcal{W}^\alpha$ factors in the above currents, i.e.

$$\begin{aligned}
0 &= -\bar{D}^2\left\langle \mathrm{tr}\left(\bar{\phi}\, e^V \frac{1}{z-\phi}\right)\right\rangle - \bar{D}^2\left\langle \mathrm{tr}\left(\bar{\phi}\, e^V \frac{1}{-z-\phi}\right)\right\rangle + \\
&\quad +2\bar{D}^2\left\langle \mathrm{tr}\left(\bar{\tilde{x}}\, e^V \frac{1}{z-\phi} x \left(\frac{1}{z+\phi}\right)^T\right)\right\rangle \\
&= 2R(z)T(z) + 2R(-z)T(-z) + R(z)T(-z) + R(-z)T(z) - W'(z)T(z) - W'(-z)T(-z) - \\
&\quad - c_1(z) - \frac{2\beta}{z}[R(z) - R(-z)]\,,
\end{aligned} \tag{3.35}$$

where the polynomial $c_1(z)$ is explicitly given by

$$c_1(z) = -\left\langle \mathrm{tr}\left(\frac{W'(z) - W'(\phi)}{z-\phi}\right)\right\rangle - \left\langle \mathrm{tr}\left(\frac{W'(-z) - W'(\phi)}{-z-\phi}\right)\right\rangle. \tag{3.36}$$

As in section 2.2, we note that, were it not for the β-dependent terms, eq. (3.35) could be viewed as the formal derivative of eq. (3.29), provided that $c_1 = \frac{\partial}{\partial S} r_1$; $T = \frac{\partial}{\partial S} R = \frac{\partial}{\partial S}\omega_0$. To deal with the β-dependent terms we recall eq. (3.11) and note that the identification[12]

$$T(z) = \frac{\partial}{\partial S}\omega_0 + 4\omega_{1/2} \tag{3.37}$$

resolves the discrepancy. Our suggested expression (3.37) generalizes the formula proposed in ref. [18] (see also [17, 22, 3]). Consideration of cubic equations involving $T(z)$ and $R(z)$ presumably also leads to (3.37).

4. U(N) with symmetric or antisymmetric matter: II

In this section we discuss how to extend the approach in ref. [2] to the case with matter in the symmetric and antisymmetric representations. We will first give a heuristic argument and then give a more detailed argument and present an explicit sample calculation.

Going through steps similar to the ones carried out in [2], it can be shown that, for the purpose of determining the effective action, the superspace action for a field φ_R in some non-real representation R of U(N), together a field $\tilde{\varphi}_{\bar{R}}$ in the representation conjugate to R, can be rewritten as

$$\int \mathrm{d}^4x\, \mathrm{d}^2\theta \left[-\frac{1}{2}\tilde{\varphi}_{\bar{R}}(\Box + m - i\mathcal{W}_\alpha D_\alpha)\varphi_R + W_{\mathrm{tree}}(\varphi, \tilde{\varphi})\right]. \tag{4.1}$$

[12]Here $\partial/\partial S = \sum_i N_i \partial/\partial S_i$. See footnote 5.

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Following [2] we transform to momentum superspace (p_μ, π_α), where π_α is the fermionic momentum conjugate to the superspace coordinate θ^α. We write the propagator of the nth edge of a Feynman diagram in a Schwinger parameterization as [2]

$$\int_0^\infty \mathrm{d}s_n \mathrm{e}^{-s_n(p_n^2+\mathcal{W}^\alpha \pi_{n\alpha}+m)} \,. \tag{4.2}$$

Now, in standard double-line notation, the only difference (for planar diagrams drawn on the sphere) between the symmetric (or antisymmetric) representation and the adjoint one is that the orientation of one of the lines has changed.[13]. Compared to the case with adjoint fields only, this means that each insertion of a $\mathcal{W}_\alpha$ on a line with flipped orientation comes with a minus sign. However, since W_{eff} is a function of the glueball fields[14] $S_i = -\frac{1}{32\pi^2}\mathrm{tr}(\mathcal{W}_{i\alpha}\mathcal{W}_i^\alpha)$ there are necessarily an even number (zero or two) of insertions on each line (index loop) and thus the extra minus signs cancel out and as in [2, 23] one finds

$$W_{\text{eff}} = \sum_i N_i \frac{\partial}{\partial S_i} F_{S^2} + 4\, F_{\mathbb{RP}^2} \,. \tag{4.3}$$

Here the second piece arises from the twisted part of the propagator (equivalently, from planar diagrams on $\mathbb{RP}^2$). A gauge theoretic argument for the presence of this piece can be given along the lines of refs. [23]. The factor of 4 has the same origin as in the SO / Sp models discussed in [23]. Also note that the factor of 4 that appeared in (3.37) is the same as the one in the equation above, as can been seen by using the methods in ref. [22, 18].

On the other hand, the gauge-coupling matrix $\tau_{ij}(S)$ couples to $\mathrm{tr}(\mathcal{W}^i_\alpha)\mathrm{tr}(\mathcal{W}^{j\alpha})$ in the effective action and since insertions of a single $\mathcal{W}_\alpha$ on an index loop coming from one of the lines with flipped orientation leads to a sign change, this implies that τ_{ij} will in general no longer will be given by $\frac{\partial^2 F}{\partial S_i \partial S_j}$.[15] This is also clear from the point of view in [3] where it was argued that the relation between τ_{ij} and $\frac{\partial^2 F}{\partial S_i \partial S_j}$ follows from a shift symmetry of the U(1) part of the gauge superfield $\mathcal{W}_\alpha$. In the case of adjoint fields only, this symmetry is a consequence of the decoupling of the U(1) (since the adjoint action is via commutators). However, the symmetric (or antisymmetric) representation couples to the U(1) (the gauge action is longer via commutators) so there is no shift symmetry and hence no direct relation between τ_{ij} and $\frac{\partial^2 F}{\partial S_i \partial S_j}$.

However, even though τ_{ij} is not given by $\frac{\partial^2 F}{\partial S_i \partial S_j}$, there is a simple way to keep track of the extra signs in matrix-model perturbation theory, i.e. to determine τ_{ij} perturbatively from the matrix model. To demonstrate this, we represent the $\tilde{X}\,\Phi\,X$ matrix-model vertex graphically as in figure 1. Now if one proceeds to calculate the matrix-model Feynman diagrams as usual, but for each index-loop constructed from a dashed line one makes the

[13] In addition the propagator for fields in the symmetric (or antisymmetric) representation has a twisted part not present for adjoint fields. This feature leads to the presence of planar diagrams drawn on $\mathbb{RP}^2$ in the topological expansion.

[14] For simplicity we restrict to the case of only one glueball field, $S = -\frac{1}{32\pi^2}\mathrm{tr}(\mathcal{W}_\alpha\mathcal{W}^\alpha)$, in our explicit calculations, but whenever possible we write the formulæ in their general form.

[15] A similar discrepancy was observed in [24] in the context of multi-trace operators, but in that case the discrepancy appeared already in W_{eff}.

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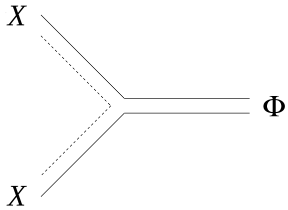

Figure 1: The $\tilde{X}\,\Phi\,X$-vertex. The dashed line has the opposite orientation compared to the Φ^3 vertex.

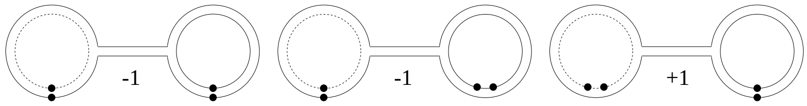

Figure 2: The black dots indicate $\mathcal{W}_\alpha$ insertions.

identification $g_{\rm s} M_i = \tilde{S}_i$, one will obtain a free energy of the form $F(S, \tilde{S})$. If one then takes the second derivative $\frac{\partial^2 F(S,\tilde{S})}{\partial S_i \partial S_j}$ using the rule $\frac{\partial \tilde{S}_i}{\partial S_j} = -\delta_i^j$, and then afterwards sets $\tilde{S}_i = S_i$, one will obtain the right result for τ_{ij}. That is, the extra signs will be taken care of and the resulting τ_{ij} will agree with the gauge theory result.

We will now give more details for a specific set of diagrams (for simplicity we consider the case of a single glueball field, S). We consider the three gauge theory diagrams in figure 2.

There are two Schwinger parameters corresponding to the momentum running in the two loops. Calling these s_1, s_2, the integral over bosonic momenta gives const $\times\ (s_1\, s_2)^{-2}$. The integral over fermionic momenta gives for the sum of the above three diagrams (plus an additional $\mathbb{RP}^2$ diagram not shown in the figure above) a constant times

$$\begin{aligned}
(s_1\, s_2)^2 (\mathcal{W}_\alpha)_j{}^i (\mathcal{W}^\alpha)_l{}^k (\mathcal{W}_\beta)_n{}^m (\mathcal{W}^\beta)_q{}^p (T_i{}^j)_r{}^s{}_t{}^u (T_k{}^l)_u{}^t{}_s{}^v (T_m{}^n)^{bc}_{va} (T_p{}^q)^{ar}_{cb} = \\
= (s_1 s_2)^2 [3N \mathrm{tr}(\mathcal{W}_\alpha \mathcal{W}^\alpha) \mathrm{tr}(\mathcal{W}_\beta \mathcal{W}^\beta) + 4\beta \mathrm{tr}(\mathcal{W}_\alpha \mathcal{W}^\alpha) \mathrm{tr}(\mathcal{W}_\beta \mathcal{W}^\beta) + \\
+ 2 \mathrm{tr}(\mathcal{W}_\beta \mathcal{W}^\beta) \mathrm{tr}(\mathcal{W}_\alpha) \mathrm{tr}(\mathcal{W}^\alpha)] \\
\propto (s_1 s_2)^2 [N(3S^2) + 4\beta S^2 - \ S\, w_\alpha w^\alpha]\,,
\end{aligned} \tag{4.4}$$

where we have used the formulæ in appendix B as well as the definition $w_\alpha = \frac{1}{4\pi}\mathrm{tr}(W_\alpha)$. We see that the s_i dependence cancels between the bosonic and fermionic momentum integrals as required for the reduction to a matrix model. For comparison, if all the fields had been in the adjoint representation, one would have obtained instead the same constant as above times

$$\begin{aligned}
(s_1\, s_2)^2 (\mathcal{W}_\alpha)_j{}^i (\mathcal{W}^\alpha)_l{}^k (\mathcal{W}_\beta)_n{}^m (\mathcal{W}^\beta)_q{}^p (T_i{}^j)_r{}^s{}_t{}^u (T_k{}^l)_u{}^t{}_s{}^v (T_m{}^n)_v{}^a{}_b{}^c (T_p{}^q)_c{}^b{}_a{}^r = \\
= (s_1 s_2)^2 [3N \mathrm{tr}(\mathcal{W}_\alpha \mathcal{W}^\alpha) \mathrm{tr}(\mathcal{W}_\beta \mathcal{W}^\beta) - 6 \mathrm{tr}(\mathcal{W}_\beta \mathcal{W}^\beta) \mathrm{tr}(\mathcal{W}_\alpha) \mathrm{tr}(\mathcal{W}^\alpha)] \\
\propto (s_1 s_2)^2 [N(3S^2) + 3\, S\, w_\alpha w^\alpha]\,.
\end{aligned} \tag{4.5}$$

By comparing (4.4) and (4.5) we see that the terms proportional to N agree. These are contributions to $N \frac{\partial}{\partial S} F_{S^2}$. The second term in (4.4) contributes to $4F_{\mathbb{RP}^2}$ and comes from a

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diagram (not displayed in the figure above) with a twisted propagator. Finally, the last set of terms contribute to $\frac{1}{2}\tau(s)w_\alpha w^\alpha$ and explicitly illustrate the sign rule discussed above. In the first case we have $-1-1+1=-1$, and in the second case we get $+1+1+1=+3$. The relative factor of $-\frac{1}{3}$ is indeed present in (4.4) vs. (4.5).

5. Summary

In this paper we focused on three $\mathcal{N}=1$ supersymmetric gauge theories: $\mathrm{U}(N)\times\mathrm{U}(N)$ with matter in adjoint and bifundamental representations, $\mathrm{U}(N)$ with matter in adjoint and symmetric representations, and $\mathrm{U}(N)$ with matter in adjoint and antisymmetric representations. As was shown, each of these theories exhibits a cubic algebraic curve. The equivalence of the matrix model and the gauge theory descriptions was established by means of generalized Konishi anomalies equations, which were shown to be equivalent to the matrix model loop equations. This result demonstrates the equivalence of the matrix models to the holomorphic sector of the gauge theories.

In addition, we studied the relation between the generating functions $T(z)$ of gauge theory vevs and the generating functions $\omega(z)$ of matrix model vevs for each of the theories considered, generalizing the results of refs. [17, 18].

We also investigated the matrix model/gauge theory equivalence using a perturbative superspace analysis. If matter in the symmetric (or antisymmetric) representation is present it was shown that the gauge-coupling matrix τ_{ij} is *not* given by the second derivative of the matrix model free energy; the latter must be modified diagram-by-diagram by suitably chosen minus signs in the matrix-model perturbative expansion so as to yield the correct gauge coupling matrix. As a result there does not appear at this point to be a concise formula expressing τ_{ij} in terms of the matrix model free energy, contrary to situations involving only adjoint, fundamental, or bifundamental matter.

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A. Saddle-point analysis

Here we will briefly discuss the saddle-point approach to study the planar solution of the matrix models discussed in this paper. In the recent papers [11, 12] this approach has been extended to holomorphic matrix models.

A.1 $\mathrm{U}(N)\times\mathrm{U}(N)$ with bifundamental matter

The saddle-point approach to the $\mathrm{U}(N)\times\mathrm{U}(N)$ quiver model has previously been discussed in refs. [14, 15, 8, 10] (and was recently extended to holomorphic matrices in [11]). The first step is to transform to an eigenvalue basis for the adjoint fields and then integrate out

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the bifundamental fields. This reduces (2.2) to

$$Z \propto \int \prod_i \mathrm{d}\lambda_i \,\mathrm{d}\tilde{\lambda}_i \frac{\prod_{i<j}(\lambda_i-\lambda_j)^2(\tilde{\lambda}_i-\tilde{\lambda}_j)^2}{\prod_{i,j}(\lambda_i-\tilde{\lambda}_j)} \mathrm{e}^{-\frac{1}{g_s}\sum_i\left[W(\lambda_i)-\tilde{W}(\tilde{\lambda}_i)\right]}. \tag{A.1}$$

The saddle-point equations of motion are thus

$$\begin{aligned} -\frac{W'(\lambda_i)}{g_s} + 2\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j} - \sum_j \frac{1}{\lambda_i-\tilde{\lambda}_j} &= 0\,, \\ \frac{\tilde{W}'(\tilde{\lambda}_i)}{g_s} + 2\sum_{j\neq i}\frac{1}{\tilde{\lambda}_i-\tilde{\lambda}_j} - \sum_j \frac{1}{\tilde{\lambda}_i-\lambda_j} &= 0\,. \end{aligned} \tag{A.2}$$

The equations (2.12) and (2.23) can be derived directly from the saddle-point equations. For instance, (2.12) can be obtained by considering

$$\begin{aligned} &g_{\mathrm{s}}^2\sum_i \frac{1}{z-\lambda_i}\left[-\frac{W'(\lambda_i)}{g_s} + 2\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j} - \sum_j \frac{1}{\lambda_i-\tilde{\lambda}_j}\right] + \\ &+g_{\mathrm{s}}^2\sum_i \frac{1}{z-\tilde{\lambda}_i}\left[\frac{\tilde{W}'(\tilde{\lambda}_i)}{g_s} + 2\sum_{j\neq i}\frac{1}{\tilde{\lambda}_i-\tilde{\lambda}_j} - \sum_j \frac{1}{\tilde{\lambda}_i-\lambda_j}\right] = 0\,. \end{aligned} \tag{A.3}$$

Alternatively, an expedient way to obtain (2.12) and (2.23) is by imposing [15, 8, 10]

$$\oint \mathrm{d}z \frac{1}{x-z} W^{(s)}(z) = 0\,, \tag{A.4}$$

where the contour encloses all eigenvalues but not the point x and the W-algebra currents $W^{(s)}(z)$ $(s=2,3)$ are given by

$$W^{(s)} = \frac{(-1)^s}{s}\sum_{i=1}^{3}(u_i)^s\,, \tag{A.5}$$

where the u_i's were defined in (2.14) and below (2.19).

A.2 U(N) with (anti)symmetric matter

The first step of the saddle-point approach is to transform (3.2) into an eigenvalue basis for Φ and then integrate out x_{ij} and $\tilde{x}^{ij}$. This leads to

$$Z \propto \int \prod_i \mathrm{d}\lambda_i \frac{\prod_{i<j}(\lambda_i-\lambda_j)^2 \prod_i \lambda_i^{-\beta/2}}{\prod_{i,j}(\lambda_i+\lambda_j)^{1/2}} \mathrm{e}^{-\frac{1}{g_s}\sum_i W(\lambda_i)}\,. \tag{A.6}$$

The saddle-point equation of motion is thus

$$-\frac{W'(\lambda_i)}{g_s} + 2\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j} - \sum_j\frac{1}{\lambda_i+\lambda_j} - \frac{\beta}{2}\frac{1}{\lambda_i} = 0\,, \tag{A.7}$$

where the last term is a $1/M$ (or g_{s}) effect.

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The model (A.6) is closely related to the O(n) matrix model [19] with $n = 1$. The planar solution of that model was derived in [20]. (The extension to holomorphic matrices was recently discussed in ref. [12].)

The expressions (3.8), (3.21) can be derived directly from the saddle-point equations. For instance, (3.8) can be obtained by considering

$$\sum_i \frac{1}{z-\lambda_i}\left[-\frac{W'(\lambda_i)}{g_s}+2\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}-\sum_j\frac{1}{\lambda_i+\lambda_j}-\frac{\beta}{2}\frac{1}{\lambda_i}\right]+(z\leftrightarrow -z)=0\,. \tag{A.8}$$

Alternatively, an expedient way to obtain (3.8) and (3.21) is by imposing

$$\oint \mathrm{d}z\frac{1}{x-z}W^{(s)}(z)=0\,, \tag{A.9}$$

where the contour encloses all eigenvalues but not the point x and the W-algebra currents $W^{(s)}(z)$ $(s = 2, 3)$ are given by

$$W^{(s)}=\frac{(-1)^s}{s}\sum_{i=1}^{3}(u_i)^s\,, \tag{A.10}$$

where the u_i's were defined in (3.12) and below (3.17).

B. Some representation theory

Here we collect some explicit formulæ for the generators in the various representations discussed in the main text.

Adjoint representation of U(N)**.** In standard double-index notation the generators in the adjoint representation are

$$(T_i{}^j)_k{}^l{}_m{}^n=\delta_i^n\delta_k^j\delta_m^l-\delta_i^l\delta_m^j\delta_k^n\,. \tag{B.1}$$

This gives the well-known results

$$(V\phi)_k{}^l=V_j{}^i(T_i{}^j)_k{}^l{}_m{}^n\phi_n{}^m=[V,\phi]_k{}^l\,;\qquad (\mathcal{W}_\alpha\phi)_k{}^l=(\mathcal{W}_\alpha)_j{}^i(T_i{}^j)_k{}^l{}_m{}^n\phi_n{}^m=[\mathcal{W}_\alpha,\phi]_k{}^l\,, \tag{B.2}$$

where V and $\mathcal{W}_\alpha$ are the vector and spinor gauge superfields, respectively.

Bifundamental representations of U$(N)\times$ U(N)**.** To describe the action of the gauge superfields on the bifundamental field $b_i{}^{\tilde{\jmath}}$ it is convenient to use a composite index $I = (i,\tilde{\imath})$. In this notation the gauge vector superfield is $\mathsf{V} = V_J{}^I(T_I{}^J)$, where

$$(T_I{}^J)_i{}^{\tilde{\jmath}}{}_{\tilde{k}}{}^l=\delta_I^l\delta_i^J\delta_{\tilde{k}}^{\tilde{\jmath}}-\delta_I^{\tilde{\jmath}}\delta_{\tilde{k}}^J\delta_i^l\,, \tag{B.3}$$

and we have used the double-index notation. This implies

$$(\mathsf{V}b)_i{}^{\tilde{\jmath}}=V_J{}^I(T_I{}^J)_i{}^{\tilde{\jmath}}{}_{\tilde{k}}{}^l b_l{}^{\tilde{k}}=V_i{}^k b_k{}^{\tilde{\jmath}}-b_i{}^{\tilde{k}}\tilde{V}_{\tilde{k}}{}^{\tilde{\jmath}}\,, \tag{B.4}$$

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where V, $\tilde{V}$ are the gauge superfields for the two $\mathrm{U}(N)$ factors. One may view V as a diagonal $2{\times}2$ matrix, $\mathrm{diag}(V,\tilde{V})$. In this notation b and $\tilde{b}$ can be combined into an off-diagonal $2{\times}2$ matrix, and ϕ and $\tilde{\phi}$ can be combined into a diagonal $2{\times}2$ matrix. In the $2{\times}2$ matrix notation, the gauge action is via commutators. Similarly the action of the gauge spinor superfield can be written as

$$(\mathsf{W}_\alpha b)_i{}^{\tilde{j}} = (\mathcal{W}_\alpha)_i{}^k b_k{}^{\tilde{j}} - b_i{}^{\tilde{k}} (\tilde{\mathcal{W}}_\alpha)_{\tilde{k}}{}^{\tilde{j}} \,, \tag{B.5}$$

where $\mathcal{W}_\alpha$ and $\tilde{\mathcal{W}}_\alpha$ are the gauge spinor superfields corresponding to the two $\mathrm{U}(N)$ factors.

The action on the bifundamental field $\tilde{b}_{\tilde{i}}{}^j$ is the same as the one on b, but with tilde and un-tilde indices interchanged.

Symmetric/antisymmetric representation of $\mathrm{U}(N)$**.** The generators in the symmetric or antisymmetric representation of $\mathrm{U}(N)$ are

$$(T_i{}^j)_{kl}^{mn} = 2\delta^j_{[k}\delta^{[m}_{l)}\delta^{n)}_i \,, \tag{B.6}$$

where we have used the notation $u_{[i}v_{j)} = \frac{1}{2}(u_i v_j + \beta\, u_j v_i)$, where $\beta = +1$ for the symmetric representation and $\beta = -1$ for the antisymmetric representation. The action of $\mathcal{W}_\alpha$ on x is

$$(\mathcal{W}_\alpha x)_{kl} = (\mathcal{W}_\alpha)_j{}^i (T_i{}^j)_{kl}^{mn} x_{nm} = [(\mathcal{W}_\alpha)_k{}^j x_{jl} + \beta(\mathcal{W}_\alpha)_l{}^j x_{jk}] \,, \tag{B.7}$$

or in matrix notation: $\mathcal{W}_\alpha x + x(\mathcal{W}_\alpha)^T$.

The generators of semi-simple Lie algebras satisfy $\mathrm{tr}_R(T^A T^B) = I(R)\delta^{AB}$ where $I(R)$ is the index of the representation, i.e. $2N$ for the adjoint, $N-2$ for the antisymmetric and $N+2$ for the symmetric representation. Using the above forms of the generators the expression $\mathrm{tr}_R(T^A T^B)$ will contain some extra trace factors since we are dealing with $\mathrm{U}(N)$'s.

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